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Unique Fixed Point Theorems In Complete b_2-Metric Spaces

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Abstract

This paper studies some fixed point theorems for mappings in the setting of b_2-metric space. These theorems are first introduced by Venkata (2001) for mappings in the setting of 2-metric space. The theorems are referred to in this paper as theorem 5.1 on page 4 and theorem 5.2 on page 6 when are proved on b_2-metric space. In theorem 5.1, if $\gamma, \beta = 0$ and all conditions have not changed, then theorem 5.1 is still applicable. If $\gamma, \beta = 0$ in the first condition of the theorem 5.2, then theorem 5.2 is still applicable.

Keywords and phrases: fixed point, Common fixed point, 2-metric space, b_2-metric space, complete b_2-metric space.

الملخص

تدرس هذه الورقة البحثية بعض نظريات النقطة الثابتة للدوال في الفراغ b_2- المتري. وكان فينتاكا (2001) أول من قدم النظريات المدروسة في هذه الورقة على الفراغ 2-المتري. وقد اشرنا إلى هذه النظريات في هذه الورقة بالنظرية 5.1 في الصفحة 4 والنظرية 5.2 في الصفحة 6 عندما نجح إثباتها كذلك على الفراغ b_2- المتري. ففي النظرية 5.1 إذا كانت $\gamma, \beta = 0$ وجميع شروط النظرية لم تتغير فإن النظرية لا تزال قابلة للتطبيق، وفي النظرية 5.2 إذا وضعنا في الشرط الأول $\gamma, \beta = 0$ فإن النظرية لا تزال قابلة للتطبيق.

كلمات وعبارات أساسية : نقطة ثابتة، نقطة ثابتة مشتركة ، الفراغ 2- المتري، الفراغ b_2- المتري، الفراغ b_2- المتري التام.

1. Introduction

Metric space, as it is known, is a set with structure determined by a well-defined notation of distance. The idea of metric space has been generalized in many directions in mathematics.

Derived from metric space is the notation of b-metric space which was first initiated by Bourbaki (1974) and Bakhtin (1989,30). Czerwik (1993,1) gave an axiom which was weaker than the triangular inequality and formally defined a b-metric space with a view of generalizing the Banach contraction mapping theorem. On the other hand, the notion of a 2-metric space was introduced by Gähler (1963,26) having the area of a triangle in R^2 as the inspirative example.

There are many fixed point results were obtained for single and multivalued mappings in b-metric space Czerwik (1993,1) & (1998,46) and many other authors. Similarly, several fixed point results were obtained for mappings in 2-metric spaces.

The aim of this paper is to extend some fixed point theorems in 2-metric space to b_2-metric space.

2. Methodology

This paper follows a qualitative method. This is an analysis to some fixed point theories. This method is suitable for this research because we already have proven fixed point theories that are to be applied on mappings in b_2-metric space.

Data is collected from some already published literature on this topic and from the results of applying the fixed point theorems on mappings in 2-metric spaces.

3. Significance of the research

This research is significant in pure mathematics, particularly in functional analysis, because it has proven that some fixed point theorems in 2-metric spaces can be also proven on 2-metric spaces.

4. Preliminaries

Before stating our main results, some necessary definitions might be introduced as follows.

Definition 4.1. Let X be a nonempty set and $T: X \to X$ a self-map. We say that $x \in X$ is a fixed point of T if $T(x) = x$ denote by FT or $Fix(T)$ the of all fixed points of T.

Definition 4.2. For $n \geq 2$, let $f_1, f_2, \dots, f_n$ are functions from X into itself. If there exists an element x in X such that $f_1(x) = f_2(x) = \cdots = f_n(x) = x$. Then x is called a common fixed point of $f_{1,} f_2, \dots, f_n$.

Definition 4.3. Czerwik (1993, 1) & (1998, 46) Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d: X \times X \longrightarrow R^+$ is a b-metric on X if for all $x, y, z \in X$, the following conditions hold:

1) $d(x,y) = 0$ if and only if $x = y$.
2) $d(x,y) = d(y,x)$.
3) $d(x,y) \leq s[d(x,y) + d(y,z)]$.

In this case, the pair (X,d) is called a b-metric space.

Definition 4.4. Piao (2008, 24) & (2012, 3) Let X be a nonempty set and $d: X \times X \times X \longrightarrow R$ be a map satisfying the following conditions:

1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x,y,z) \neq 0$.
2) If at least two of three points x, y, z are the same, then $d(x,y,z) = 0$.
3) The symmetry:
$d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x)$ for all $x, y, z \in X$.
4) The rectangle inequality: $d(x,y,z) \leq d(x,y,a) + d(y,z,a) + d(z,x,a)$ for all $x, y, z, a \in X$.

Then d is called a 2-metric on X and (X,d) is called a 2-metric space.

Definition 4.5. Mustafa (2014, 144)Let X be a nonempty set, $s \geq 1$ be a real number and $d: X \times X \times X \longrightarrow R$ be a map satisfying the following conditions:

1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x,y,z) \neq 0$.
2) If at least two of three points x, y, z are the same, then $d(x,y,z) = 0$.
3) The symmetry:

$d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x)$ for all $x, y, z \in X$.

4) The rectangle inequality:

$d(x,y,z) \leq s[d(x,y,a) + d(y,z,a) + d(z,x,a)]$ for all $x, y, z, a \in X$.

Then d is called a b_2-metric on X and (X,d) is called a b_2-metric space.

Remark 4.1. Obviously, for $s = 1$, b_2-metric reduces to 2-metric.

The following are some typical examples of b_2-metric spaces.

Example 4.1. Mustafa (2014,144) Let a mapping $d: R^3 \longrightarrow [0,\infty)$ be defined by

$$d(x,y,z) = min\{|x-y|, |y-z|, |z-x|\}$$

Then d is a 2-metric space on R. For arbitrary real numbers x, y, z, a.Using convexity of the function $f(x) = x^p$ on $[0,\infty)$ for $p \geq 1$, we obtain that

$$d_p(x,y,z) = [min\{|x-y|, |y-z|, |z-x|\}]^p$$

is a b_2-metric on R with $s \leq 3^{p-1}$.

Example 4.2. Mustafa (2014, 144) Let a mapping $d: R^3 \longrightarrow [0,\infty)$ be defined by

$d\,(x,y,z) = \{[xy + yz + zx]^p,$
$\quad if\ x \neq y \neq z \neq x\ 0, \qquad otherwise$

$x, y, z \in X$. Then d is a b_2-metric space , with $s \leq 3^{p-1}, p \geq 1$.

Definition 4.6. Mustafa (2014, 144) Let $\{x_n\}_{n \in N}$ be a sequence in a b_2-metric space (X,d).

1) A sequence $\{x_n\}$ is said to be b_2-convergent to $x \in X$, written as $x_n = x,$ if all $a \in X, d(x_n, x, a) = 0.$
2) A sequence $\{x_n\}$ is said to be b_2- Cauchy sequence if and only if $d(x_n, x_m, a) \to 0,$ when $n, m \to \infty$. for all $a \in X$.
3) The b_2-metric space (X, d) said to be b_2-complete if every b_2-Cauchy sequence is b_2-convergent sequence.

Definition 4.7. Mustafa (2014, 144) Let (X, d) and (X', d') be two b_2- metric spaces and let $f : X \to X'$ be a mapping . Then f is said to be b_2- continuous at a point $z \in X$ if for a given $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in X$ and $d(z, x, a) < \delta$ for all $a \in X$ imply that $d'(fz, fx, a) < \varepsilon$. The mapping f is b_2- continuous on X if it is b_2-continuous at all $z \in X$.

Proposition 4.1 Mustafa (2014, 144) Let (X, d) and (X', d') be two b_2-metric spaces . Then a mapping $f : X \to X'$ is b_2-continuous at a point $x \in X$ if and only if it is b_2-sequentially continuous at x; that is, whenever $\{x_n\}_{n\in N}$ is b_2-convergent to x, $\{f(x_n)\}_{n\in N}$ is b_2-convergent to $f(x)$.

Lemma 4.2 Zaid (2015, 5) Let (X, d) be a b_2-metric space with $s \geq 1$, and let $\{x_n\}_{n\in N}$ be a sequence in X such that the sequence $d(x_n, x_{n+1}, a)$ is not increasing. Then $d(x_i, x_j, x_k) = 0$ for all $i, j, k \in N$.

5. Main Results

Theorem 5.1. Let (X, d) be a complete b_2-metric space, $s \geq 1$.Let f be a b_2 -continuous self-map of X, satisfying

$$\begin{aligned} d^2(f(x), f(y), a) &\leq \alpha d(x, f(x), a) d(y, f(y), a) + \beta d(x, f(x), a) d(y, f(x), a) \\ &+ \gamma d(y, f(y), a) d(y, f(x), a) + \delta d(x, f(y), a) d(y, f(x), a) \end{aligned}$$

for all $x, y, z \in X$ and $\alpha, \beta, \gamma, \delta \geq 0$ with max $\{\alpha, \delta\} < 1$ and $s\alpha < 1$.Then f has a unique fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary , and let $\{x_n\}_{n\in N}$ be a sequence in X.

Define $x_1 = f(x_0),\ \ x_2 = f(x_1), \dots x_n = f(x_{n-1}),\ \ n = 1,2,\dots$

We have
$$\begin{aligned} d^2(x_1,x_2,a) &= d^2(f(x_0), f(x_1), a) \\ &\le \alpha d(x_0, f(x_0), a) d(x_1, f(x_1), a) \\ &+\beta d(x_0, f(x_0), a) d(x_1, f(x_0), a) \\ &+\gamma d(x_1, f(x_1), a) d(x_1, f(x_0), a) \\ &+\delta d(x_0, f(x_1), a) d(x_1, f(x_0), a) \\ &= \alpha d(x_0, x_1, a) d(x_1, x_2, a) + \beta d(x_0, x_1, a) d(x_1, x_1, a) \\ &+\gamma d(x_1, x_2, a) d(x_1, x_1, a) + \delta d(x_0, x_2, a) d(x_1, x_1, a) \end{aligned}$$

i.e., $d^2(x_1,x_2,a) \le \alpha d(x_0,x_1,a) d(x_1,x_2,a)$

Hence $\ d(x_1,x_2,a) \le \alpha d(x_0,x_1,a)$

Similarly
$$\begin{aligned} d(x_2,x_3,a) &\le \alpha d(x_1,x_2,a) \\ &\le \alpha^2 d(x_0,x_1,a) \\ &\vdots \end{aligned}$$

i.e.,
$$d(x_n,x_{n+1},a) \le \alpha^n d(x_0,x_1,a)$$

We claim that $\{x_n\}$ is Cauchy sequence in X.

For $m > n$, we have
$$\begin{aligned} d(x_n,x_m,a) &\le s[d(x_n,x_{n+1},a) + d(x_m,x_{n+1},a) + d(x_n,x_m,x_{n+1})] \\ &\le s[\alpha^n d(x_0,x_1,a) + \alpha^n d(x_0,x_1,x_m)] + s d(x_{n+1},x_m,a) \\ &\le s\alpha^n d(x_0,x_1,a) + s\alpha^n d(x_0,x_1,x_m) \\ &\quad + s^2[d(x_{n+2},x_{n+1},a) + d(x_{n+2},x_{n+1},x_m) + d(x_{n+2},x_m,a)] \\ &\le s\alpha^n d(x_0,x_1,a) + s\alpha^n d(x_0,x_1,x_m) \\ &\quad + s^2\alpha^{n+1} d(x_0,x_1,a) + s^2\alpha^{n+1} d(x_0,x_1,x_m) + s^2 d(x_{n+2},x_m,a) \\ &\le \left(s\alpha^n + s^2\alpha^{n+1} + s^3\alpha^{n+2} + \cdots + s^{m-n-1}\alpha^{m-2}\right) d(x_0,x_1,a) \end{aligned}$$

$$+(s\alpha^n + s^2\alpha^{n+1} + s^3\alpha^{n+2} + \cdots s^{m-n-1}\alpha^{m-2})d(x_0, x_1, x_m)$$
$$+ s^{m-n}\, d(x_{m-1}, x_m, a)$$
$$\leq (s\alpha^n + s^2\alpha^{n+1} + s^3\alpha^{n+2} + \cdots + s^{m-n-1}\alpha^{m-2})d(x_0, x_1, a)$$
$$+(s\alpha^n + s^2\alpha^{n+1} + s^3\alpha^{n+2} + \cdots + s^{m-n-1}\alpha^{m-2})d(x_0, x_1, x_m)$$
$$+ s^{m-n}\alpha^{m-1}\, d(x_0, x_1, a)$$
$$= (s\alpha^n + s^2\alpha^{n+1} + s^3\alpha^{n+2} + \cdots + s^{m-n-1}\alpha^{m-2} + s^{m-n}\alpha^{m-1})d(x_0, x_1, a)$$
$$+(s\alpha^n + s^2\alpha^{n+1} + s^3\alpha^{n+2} + \cdots s^{m-n-1}\alpha^{m-2})d(x_0, x_1, x_m)$$
$$\leq \frac{s\alpha^n}{1-s\alpha} d(x_0, x_1, a) + \frac{s\alpha^n}{1-s\alpha} d(x_0, x_1, x_m) \qquad (1)$$

As $n, m \to \infty$ in (1) , we have $d(x_n, x_m, a) \to 0$.

Thus $\{x_n\}$ is Cauchy sequence in X.

Since X is complete b_2-metric space, then there exist a point $x \in X$ such that $x_n \to x$

Now to show that x is a fixed point of f. Since $x_n \to x$ as $n \to \infty$ using continuity of f, we have $f(x) = f(x_n)$

Which implies that $x_{n+1} = f(x)$.

Thus $f(x) = x$. Hence x is a fixed point of f.

For uniqueness of x: Let x, y be distinct fixed points of f.

Then for all $a \in X$ we have $d(x, y, a) = d(f(x), f(y), a)$ and

$$d^2(x, y, a) = d^2(f(x), f(y), a)$$
$$\leq \alpha d(x, f(x), a)d(y, f(y), a) + \beta d(y, f(x), a)d(x, f(x), a)$$
$$+\gamma d(y, f(y), a)d(y, f(x), a) + \delta d(x, f(y), a)d(y, f(x), a)$$
$$\leq \alpha d(x, x, a)d(y, y, a) + \beta d(y, x, a)d(x, x, a)$$
$$+\gamma d(y, y, a)d(y, x, a) + \delta d(x, y, a)d(y, x, a)$$
$$= \delta\, d^2(x, y, a).$$

Thus $(1 - \delta)d^2(x, y, a) \leq 0$.

This implies that $\delta \geq 1$, which is a contradiction to $\delta < 1$.

Therefore the fixed points are unique.

Theorem 5.2. Let (X,d) be a complete b_2-metric space. Let f and g be two continuous mappings of X into itself, such that

i) $d^2(f(x),f(y),a) \leq$

$\alpha d(g(x),f(x),a)d(g(y),f(y),a) + \beta d(g(x),f(x),a)d(g(y),f(x),a)$

$+\gamma d(g(y),f(y),a)d(g(y),f(x),a) + \delta d(g(x),f(y),a)d(g(y),f(x),a)$

for all $x,y,a \in X$ and $\alpha,\beta,\gamma,\delta \geq 0$ with $\max\{\alpha,\delta\} = 1$ and $s\alpha < 1$.

ii) $fg = gf$, $f(x) \subset g(x)$

Then f and g having a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary, since $f(x) \subset g(x)$

We can choose $x_1 \in X$ such that

$f(x_0) = g(x_1)$, $f(x_1) = g(x_2), \dots, f(x_n) = g(x_{n+1})$, $n = 1,2,\dots$

Now , $d(g(x_{n+1}),g(x_{n+2}),a) = d(f(x_n),f(x_{n+1}),a) \quad for\ all\ a \in X$

$$\begin{aligned}
d^2(g(x_{n+1}),g(x_{n+2}),a) &= d^2(f(x_n),f(x_{n+1}),a)\\
&\leq \alpha d(g(x_n),f(x_n),a)d(g(x_{n+1}),f(x_{n+1}),a)\\
&+\beta d(g(x_n),f(x_n),a)d(g(x_{n+1}),f(x_n),a)\\
&+\gamma d(g(x_{n+1}),f(x_{n+1}),a)d(g(x_{n+1}),f(x_n),a)\\
&+\delta d(g(x_n),f(x_{n+1}),a)d(g(x_{n+1}),f(x_n),a)\\
&= \alpha d(g(x_n),g(x_{n+1}),a)d(g(x_{n+1}),g(x_{n+2}),a)\\
&+\beta d(g(x_n),g(x_{n+1}),a)d(g(x_{n+1}),g(x_{n+1}),a)\\
&+\gamma d(g(x_{n+1}),g(x_{n+2}),a)d(g(x_{n+1}),g(x_{n+1}),a)\\
&+\delta d(g(x_n),g(x_{n+2}),a)d(g(x_{n+1}),g(x_{n+1}),a)\\
&= \alpha d(g(x_n),g(x_{n+1}),a)d(g(x_{n+1}),g(x_{n+2}),a)
\end{aligned}$$

Then $d(g(x_n),g(x_{n+1}),a) \leq \alpha d(g(x_{n-1}),g(x_n),a) \quad for\ all \quad a \in X$

Hence

$$d(g(x_{n+1}),g(x_{n+2}),a) \leq \alpha^{n+1} d(g(x_0),g(x_1),a)$$

For $m > n$, we have

$d(g(x_n), g(x_m), a)$

$\leq s[d(g(x_n), g(x_m), g(x_{n+1})) + d(g(x_m), g(x_{n+1}), a) + \ d(g(x_n), g(x_{n+1}), a)]$

$\leq s[\alpha^n d(g(x_0), g(x_1), g(x_m)) + \alpha^n d(g(x_0), g(x_1), a)] + sd(g(x_{n+1}), g(x_m), a)$

$\vdots$

$\leq (s\alpha^n + s^2\alpha^{n+1} + s^3\alpha^{n+2} + \cdots + s^{m-n-1}\alpha^{m-2}) d(g(x_0), g(x_1), g(x_m))$

$+(s\alpha^n + s^2\alpha^{n+1} + s^3\alpha^{n+2} + \cdots s^{m-n-1}\alpha^{m-2})\ d(g(x_0), g(x_1), a)$

$+ s^{m-n} d(g(x_{m-1}), g(x_m), a)$

$\leq (s\alpha^n + s^2\alpha^{n+1} + s^3\alpha^{n+2} + \cdots + s^{m-n-1}\alpha^{m-2}) d(g(x_0), g(x_1), g(x_m))$

$+(s\alpha^n + s^2\alpha^{n+1} + s^3\alpha^{n+2} + \cdots s^{m-n-1}\alpha^{m-2})\ d(g(x_0), g(x_1), a)$

$+ s^{m-n}\alpha^{m-1} d(g(x_0), g(x_1), a)$

$= (s\alpha^n + s^2\alpha^{n+1} + s^3\alpha^{n+2} + \cdots + s^{m-n-1}\alpha^{m-2}) d(g(x_0), g(x_1), g(x_m))$

$+(s\alpha^n + s^2\alpha^{n+1} + s^3\alpha^{n+2} + \cdots s^{m-n-1}\alpha^{m-2} + \ s^{m-n}\alpha^{m-1})\ d(g(x_0), g(x_1), a)$

$$\leq \frac{s\alpha^n}{1-s\alpha} d(g(x_0), g(x_1), g(x_m)) + \frac{s\alpha^n}{1-s\alpha} d(g(x_0), g(x_1), a) \qquad (2)$$

As $n, m \to \infty$ in (2) , we have $d(g\ (x_n\ ), g\ (x_m\ ), a) \to 0$.

Thus $\{g(x_n)\}_{n\epsilon N}$ is Cauchy sequence in X.

Since X is complete b_2-metric space, then there exist a point $x \in X$ such that $g(x_n) \to x$

We have $g(x_n)\ = x\ = x_{n+1}\ ) =\ f(x_n)$

Since $g x_n\ ) = f g x_n\ )$ and by continuity of g, f , we have $f(x) = g\ (\ x\ )$.

To show that *x* is a fixed point of g, we have $x = g\ (\ x\ )$.

$d^2(g\ (x), x, a) = \lim_{n\to\infty} d^2(f(x), f(\ x_n), a)$

$\leq \lim_{n\to\infty} [\ \alpha d(g(x), f(x), a) d(g(x_n), f\ (\ x_n), a)$

$+\beta d(g\ (x_n), f(x), a) d(g\ (x), f(x), a)$

$+ \gamma d(g(x_n), f(x_n), a) d(g(x_n), f(x), a)$

$$+\delta d(g(x), f(x_n), a) d(g(x_n), f(x), a)]$$
$$= [\alpha d(g(x), g(x), a) d(x, x, a) + \beta d(x, g(x), a) d(g(x), g(x), a)$$
$$+\gamma d(x, x, a) d(x, g(x), a) + \delta d(g(x), x, a) d(x, g(x), a)]$$
$$= \delta d^2(g(x), x, a).$$

Thus $(1-\delta) d^2(g(x), x, a) \leq 0, \ \ \delta < 1$.

Hence $d^2(g(x), x, a) = 0$ for all $a \in X$.

This implies that $g(x) = x$. Therefore $g(x) = f(x) = x$.

Thus x is common fixed point of f and g.

For the uniqueness of the common fixed point. Let x, y be two common fixed points of f and g, so $d(x, y, a) = d(f(x), f(y), a)$. Then from (i)

$$d^2(x, y, a) = d^2(f(x), f(y), a)$$

$$\leq [\alpha d(g(x), f(x), a) d(g(y), f(y), a)$$
$$+\beta d(g(y), f(x), a) d(g(x), f(x), a)$$
$$+\gamma d(g(y), f(y), a) d(g(y), f(x), a)$$
$$+\delta d(g(x), f(y), a) d(g(y), f(x), a)]$$
$$= [\alpha d(x, x, a) d(y, y, a) + \beta d(y, x, a) d(x, x, a)$$
$$+\gamma d(y, y, a) d(y, x, a) + \delta d(x, y, a) d(y, x, a)]$$
$$= \delta d^2(x, y, a).$$

So $(1-\delta)\, d^2\, (g(x), x, a) \leq 0$ for all $a \in X, \ \ \delta < 1$.

This implies that $d^2\, (g(x), x, a) = 0$, that is $d(g(x), x, a) = 0$ for all $a \in X$,

Thus $x = y$.

6. Conclusion

In this paper, some common fixed point theorems for two mappings have been applied on complete b_2-metric space. The results have been encouraging. In theorem 5.1, if $\gamma, \beta = 0$ and all conditions have not changed, then theorem 5.1 is still applicable. If $\gamma, \beta = 0$ in the first condition of the theorem 5.2, then theorem 5.2 is still applicable.

References

Bakhtin, I. A. (1989). The contraction mapping principle in almost metric spaces, *Funct . Anal*, pp. 26-37.

Bourbaki, N. (1974). Topologies General. *Herman: Paris, France* .

Czerwik, S. (1993). Contraction mapping in b-metric spaces. *Acta Mathematica et Informatica Universitatis Ostravensis* , pp. 5-11.

Czerwik, S. (1998). Nonlinear Set-valued Contraction Mappings in b-metric spaces. *Atti del Seminario Matematico e Fisico (Dell Univ.di Modena)*, pp. 263-276.

Gähler, V.S. (1963). 2-metrische Räume und ihre topologische Struktur. *Math.Nachr* , pp. 115-118.

Mustafa, Z., Parvaech, V., Roshan, J.R. and Kadelburg Z. (2014). b_2-metric spaces and some Fixed Point Theorems.Fixed Point Theory and Applications ,pp.144.

https://doi.org/10.1186/1687-I812-2014-144

Piao, Y. and Jin, Y.F. (2012). New Unique Common Fixed Results for Mappings with Ø-Contractive Type in 2-metric spaces. *Applied Mathematics*, pp. 734-737.

https://doi.org/10.4236/am.2012.3710

Piao, Y.J. (2008). Unique Common Fixed Points for a Family of Self-Maps with Same Type Contractive Condition in 2-metric spaces. *Analysis in Theory and Applications* , pp. 316-320. https://doi.org/10.1007/s10496-008-0316-9

Piao, Y.J. (2012). Uniqueness of Common Fixed Point for a Family of Mappings with Ø-Contractive Condition in 2-metric spaces. *Applied Mathematics,* pp. 73-77.

https://doi.org/10.4236/am.2012.31012

Venkata ,S. (2001). Some Fixed Point Theorem in Metric and 2-Metric Spaces. *IJMMS* , pp. 625-636.

ijmms.hindawu.com

Zaid, M. Abd Ghafur, B., Stojan, R., Vildan,. (2015). Some Fixed Points Results of b_2-metric spaces. *Far est journal of mathematical sciences(FJMS)*, pp. 533-548.

http://dx.doi.og/10.17654/