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The Constructions of the Square Complex of a Diagram Group from a Graphical Presentation

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Abstract: In this paper, we may obtain diagram groups for any given graphical presentation. These groups can be viewed as the fundamental group of squire complexes. Let $^4S =< a_1, a_2, a_3, a_4 : a_i = a_j ; 1 \leq i < j \leq 4 >$ be a semigroup presentation. The problems are divided into several cases according to the length of words, with all vertices in ^4K_i being words of the length i. The main aim of this article is to construct the connected square complex graph ^4K_i of a diagram group from semigroup presentation ^4S. Then we will prove $^4K_{i+1}$ is the covering squire complexes for ^4K_i for all $i \in N$. Then the covering space is identified for all connected square complex graphs by picking normal subgroups from the diagram group that was previously obtained from the semigroup presentation. This research introduces how to associate H with the covering space $^4K_{H_i}$, how to determine the generators for covering space $^4K_{H_i}$, and what $^4K_{H_i}$ looks like.

إنشاء المجمع المربع لمجموعة مخططات من عرض بياني

الكلمـــــات المفتاحيــــة :
مجموعة مخططات،
عرض شبه زمرة،
مولدات،
الشجرة الجزئية العظمى.

المستخلص : في هذه الورقة، لاي عرض بياني، قد نحصل على مجموعة مخططات. هذه المخططات يمكن النظر اليها على أنها المجموعة الأساسية للمجمعات المربعة. ليكن $^4S =< a_1, a_2, a_3, a_4 : a_i = a_j ; 1 \leq i < j \leq 4 >$ عرض شبه زمرة. نقسم المشكلة الى عدة حالات وفقاً لطول المسار مع جعل جميع القمم في ^4K_i هي كلمات ذات طول i. . الهدف الرئيسي من هذه المقالة هو إنشاء بيان متصل مربع ^4K_i من شبه الزمرة ^4S. ثم سنثبت أن K_{i+1} هو غطاء ل ^4K_i لأجل كل $i \in N$. ومن ثم تحديد مساحة الغطاء لجميع البيانات المتصلة المربعة عن طريق اختيار زمرة جزئية ناظمية من المخطط التي تم الحصول عليه سابقاً من ^4S. نقدم كذلك كيفية ربط H مع الغطاء $^4K_{H_i}$ وكيف يمكن إيجاد مولدات $^4K_{H_i}$ وكيف يبدو شكل $^4K_{H_i}$.

INTRODUCTION

The first definition of diagram groups was introduced by (Meakin & Sapir, 1993); however, their student, (Kilibarda, 1994, 1997), had worked out the first result on a diagram group. Her work proved that every equivalence class of semigroup diagrams contains a unique diagram without dipoles. Such diagrams are called ‘reduced’. Further results about diagram groups were discussed in the published work of (Guba & Sapir, 1997; Guba & Sapir, 2006a, 2006b; Guba, 2002; Guba & Sapir, 1999; Guba & Sapir, 2002) presented an equivalent complex to K(S) and also referred to it as the Squire complex. Monoid pictures were studied by (Pride, 1991, 1993, 1995), whereas the notion of the pictures was attributed to (Guba, 2002). Guba used transistors with top label and bottom label and straight vertical wires, while Pride used a circle with two distinguished base points (Pride, 1993) or one distinguished base point (Pride, 1991) and arbitrary curves, respectively. (Nieveen & Smith, 2006)

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discussed the 'covering spaces' and subgroups of free groups. (Gheisari & Ahmad, 2010a, 2010b) managed to obtain the generators and the spanning tree in graphs from diagram groups over semigroup presentation using the lifting method.

Diagram groups are one form of geometrical objects called "semigroup diagrams". Each diagram group is determined by an alphabet X, containing all possible labels of edges, a set of relations $r = \{U_i = V_i,\ i = 1,2,\dots\}$, containing all possible labels of cells, and a word W over X- the label of the top and bottom paths of diagrams. Diagrams can be considered 2-dimensional words, and diagram groups can be considered a square dimensional analogue of a free group.

If a group is representable by diagrams (that is, it is a subgroup of a diagram group), then one can use the geometry of planar graphs to deduce certain properties of the group. (Guba & Sapir, 1997) viewed diagrams as 2-dimensional words; they developed a calculus called combinatorics on diagrams. The geometry of diagrams allows one to consider many homomorphismk from diagram groups into the group of piecewise linear homeomorphisms of the real line. Thus, a connection between groups is represented by diagrams, and groups are represented by piecewise linear functions. This connection can be used in both directions.

Let $S =< X: r >$ be a semigroup presentation. Then the diagram group $D(S, W)$ can be obtained, where W is a positive word on X, as given by (Guba & Sapir, 1997). The square complex, associated with semigroup presentation S, is denoted by $K(S)$ with a binary operation, $[\alpha].[\beta] = [\alpha\beta]$, and forms a group named the 'fundamental group' with the basepoint W, denoted by $\pi_1(K(S), W)$, where α, β are two closed paths. (Kilibarda, 1994, 1997) proved that every diagram group over semigroup presentation S is isomorphic to the fundamental group of 2-complex associated with this presentation. It will be demonstrated in this article that the square complex K obtained from S is actually a union of K_i, where K_i contains all vertices of length i.

As with square complexes, it is possible to obtain all connected square complex graphs ${}^4K_i$ for all $i \in N$ from semigroup presentation ${}^4S =< a_1, a_2, a_3, a_4: a_i = a_j; 1 \le i < j \le 4 >$, depending on the length of the words. Then the projection mapping between ${}^4K_i$ and ${}^4K_{i+1}$ can be obtained. It is important to note that any square complex will contain vertices, edges, and 2-cells. A square complex without 2-cells is simply a graph. From those semigroup presentations the covering space for all connected square complex graphs ${}^4K_i$, $i \in N$ can be determined by picking groups from the resulting diagram group.

As with a group, it is sufficient to determine the group's generators. These generators can be obtained from the square complex by identifying the maximal subtree T. Fix a vertex v, where v belongs to square complex graph ${}^4K_i$, $i \in N$, and let e be an edge such that $e \notin T$, then $\gamma_{i(e)}e(\gamma_{\tau(e)})^{-1}$ is the generator, where $\gamma_{i(e)}, \gamma_{\tau(e)}$ are paths in the maximal subtree T from $v \in {}^4K_i$, $i \in N$, to the initial and terminal of e, respectively.

PRELIMINARIES

In this section, we introduce some concepts, terminologies, and theorems, such as semigroup presentation, graphs, and square complexes that are necessary to highlights.

Definition 1: Let X be the set of alphabets. A semigroup presentation S is a pair $< X: r >$, where $r \subseteq X \times X$. An element $x \in X$ is called a 'generating symbol'; while an element $(U, V) \in r$ is called a 'defining relation', and is usually written as $U = V$. The semigroup defined by a presentation is $X^+/_{\approx}$, where $\approx$ is the smallest congruence on X^+ containing r. More generally, a semigroup S is said to be defined by the presentation $< X: r >$ if

$S \cong X^+/_{\approx}$. Thus, elements of S are in one-one correspondence with congruence classes of words from X^+ representing elements of S. For the sake of simplicity, it will be always assumed that the set of relations r in every semigroup presentation $S =< X: r >$ satisfies the following condition: if $(U, V) \in r$, then $(V, U) \notin r$.

Definition 2: A graph Γ consists of five pairs $(E, V, i, \tau, -1)$ where V and E are two disjoint finite sets. Set V is known as the set of vertices; while E as the set of edges. Symbols $i, \tau, -1$ are functions:
$i: E \longrightarrow V$, $\tau: E \longrightarrow V$, $-1: E \longrightarrow V$
such that:
$i(e) = \tau(e^{-1}), \tau(e) = i(e^{-1}), e \neq e^{-1} \ \forall e \in E$.
If e is an edge, then $i(e)$ is called the 'initial vertex' of e, and $\tau(e)$ is called the 'terminal vertex' of e.

Definition 3: A graph Γ is connected if and only if $\forall(v_1, v_2) \in V$, then there exists a path γ, such that $i(\gamma) = v_1$ and $\tau(\gamma) = v_2$. That means a graph Γ is said to be connected if any two vertices can be joined by a path.

Theorem 1 (Cohen, 1989; Rotman, 1995; Serre & Serre, 1980) : Let K be a connected 2-complex and fix a vertex v. The algebraic system
$\pi_1(K, v) = \{[\alpha] : i(\alpha) = \tau(\alpha) = v\}$.
with binary operation $[\alpha] \cdot [\beta] = [\alpha\beta]$ forms a group named the first fundamental group of K with base point v. The identity is $[1_v]$, while the inverse $[\alpha]^{-1} = [\alpha^{-1}]$.

Definition 4: A tree T in Γ is a connected graph without a cycle (loop). If a tree T contains all vertices of graph Γ, then T is called a maximal subtree or spanning tree. Geodesic in a tree is a path without backtracking, that is if γ and λ belong to Γ are two paths such that $i(\gamma) = i(\lambda)$ and $\tau(\gamma) = \tau(\lambda)$, then $\gamma = \lambda$..

Definition 5: A square complex K is a pair $< \Gamma: \mathfrak{R} >$, where Γ is a graph and $\mathfrak{R}$ is a set of cyclically reduced closed paths in Γ. Set Γ is called the skeleton of Γ and denoted by $K^{(1)}$, also the elements of r are called defining paths. This square complex is finite if Γ is finite, and it is connected if Γ is connected.

Definition 6: Let $K' =< \Gamma': \mathfrak{R}' >$ and $K =< \Gamma: \mathfrak{R} >$ be square complex graphs. A mapping $\psi : K' \to K$ is a mapping of square complexes graph from Γ' to Γ, such that $\psi(\rho) \in \Gamma$ for each $\rho \in \Gamma'$.

Definition 7: Let $\psi : K' =< \Gamma': \mathfrak{R}' > \to K =< \Gamma: \mathfrak{R} >$ be a mapping of square complexes. Then, ψ is said to be locally bijective if:

i. It is locally bijective of graphs.
ii. Γ consists of all the lifts of elements of $\mathfrak{R}$. (Note, in particular, that all lifts of elements of $\mathfrak{R}$ must be closed).

Theorem 2 (Rotman, 2002; Rotman, 1995): Let $\psi : K' \to K$ be a mapping of square complexes graphs. If v is a vertex of K such that $\psi(\tilde{v}) = v$, then $\tilde{v}$ is said to lie over v. Let α be a path in K with $i(\alpha) = v$ and suppose $\tilde{v}$ lies over v. A path $\tilde{\alpha}$ in K' is said to be a lift of α at $\tilde{v}$ if $\psi(\tilde{\alpha}) = \alpha$.

Theorem 3 (Rotman, 2002; Rotman, 1995): Let $\psi : K' \to K$ be a mapping of square complexes graphs. Then the following are equivalent:

i. The map ψ is locally injective.
ii. For each path α in K, if $\tilde{v}$ lies over $i(\alpha)$, then α has at most one lift at $\tilde{v}$.

Theorem 4 (Rotman, 2002; Rotman, 1995): Let $\psi : K' \to K$ be a mapping of square complexes graphs. Then the following are equivalent:

i. The map ψ is locally surjective.
ii. For each path α in K, if $\tilde{v}$ lies over $i(\alpha)$, then α has at least one lift at $\tilde{v}$.

Definition 8: If $\psi : K' \to K$ is a locally bijective map, then K' is called a covering complex (**covering space**) of K. The mapping ψ is called the covering map (covering projection).

Definition 9: Let $S = < X: r >$ be a semigroup presentation and U is a word on X, then

i. If $U = a$ then, $\varepsilon(a)$ denotes the plane gph consisting of one edge with the initial (terminal) vertex coinciding with the initial (terminal) vertex of the edge, as shown in Figure 1.

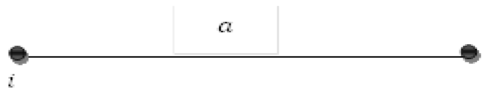

Figure (1). Graph $\varepsilon(a)$

ii. If $U = a_1 a_2 \dots a_n$, then $\varepsilon(U) = \varepsilon(a_1) + \varepsilon(a_2) + \cdots + \varepsilon(a_n)$ is called the trivial (U, U) −diagram, and the plane graph as in Figure 2.

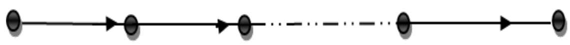

Figure (2). Trivial (U, U) −diagram

iii. If $U = 1$ the empty diagram is denoted $\varepsilon(1)$.

Definition 10: Let U and V be positive words. Let us take the graphs $\varepsilon(U)$ and $\varepsilon(V)$ such that $i(\varepsilon(U)) = i(\varepsilon(V))$ and $\tau(\varepsilon(U)) = \tau(\varepsilon(V))$ and satisfy the following condition:

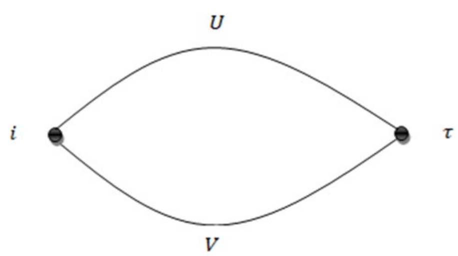

Figure (3). Elementary (U,V)-diagram

Definition 11: Let $M = N$ be a relation and let $(U, M \to N, V)$ be an edge of Γ (i.e. rewriting system). Then, the graph $\varepsilon(U) + \psi_{M,N} + \varepsilon(V)$ is called the atomic diagram, corresponding to an edge given by Figure 4.

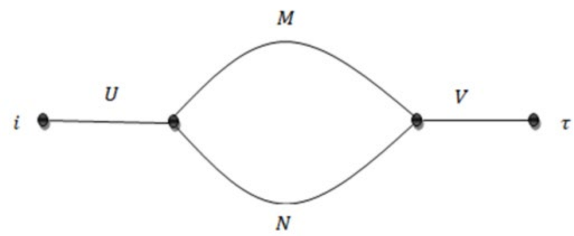

Figure (4). Atomic diagram

Definition 12: Any plane graph which is either equal to $\varepsilon(U)$ for some word U or is a concatenation of atomic diagram is called **a semigroup diagram**. If the label of the top path of a diagram is U and the label of the bottom path is V, then the diagram is called (U, V) −diagram.

Definition 13:(Rotman, 2002; Rotman, 1995) Let $\Omega : K' \to K$ be a mapping of graphs and $v' \in V'$. Consider $\mathrm{star}(v')$, suppose that we break up $\mathrm{star}(v')$ to two component where $\mathrm{star}\,(v') = v$

$\mathrm{star}\,(v') = \mathrm{star}(v^*) \cup \mathrm{star}(\tilde{v})$

such that

$\mathrm{star}\,(v^*) = \{e \in \mathrm{star}\,(v') : \Omega(e) \neq 1_v\}$

and

$\mathrm{star}\,(\tilde{v}) = \{e \in \mathrm{star}\,(v') : \Omega(e) = 1_v\}$.

So, we define here Ω is a locally injective on $\mathrm{star}(v')$ if it is an injective on $\mathrm{star}\,(v^*)$. Note that if $\mathrm{star}\,(v') = \emptyset$ and Ω is a locally injective, then clearly Ω is injective.

Definition 14: Let $\Omega : K' \to K$ be a mapping of square complexes graphs. If $\tilde{v}$ is a vertex in K', then there is induced monomorphism

$$\Omega^*: \pi_1(K', v') \to \pi_1(K, \Omega(v'))$$

defined by $\Omega^*[\alpha'] = [\Omega(\alpha')]$. The mapping Ω^* is an injective if Ω is a locally bijective. $\mathrm{Im}\,\Omega^*_H = H$

Let H be a subgroup of $\pi_1(K(S), W)$ fix a vertex O in the connected square complex graph. Now, the covering space ${}^4K_{H_i}$ will be constructed and then to obtain the covering map Ω_H: ${}^4K_{H_i} \rightarrow {}^4K_i$ in a similar way. Let v be a vertex of ${}^4K_i$ and consider the collection of paths

$$P_v = \{ [\alpha]: i(\alpha) = 0, \tau(\alpha) = v\}.$$

3 Constructions Of The General Square Complex Of Diagram Group

In this section we obtain the general method to determine the connected square complex of the diagram group over semigroup presentation ${}^4S =< a_1, a_2, a_3, a_4: a_i = a_j; 1 \leq i < j \leq 4 >$. Note that the connected square complex graph from ${}^4S$ is collections of subgraphs. The square complex graph ${}^4K_i, i \in N$ obtained from ${}^4S$ is a union of ${}^4K_i$ connecting all vertices of length i and respective edges.

For example, in Figure 5, the connected square complex graph ${}^4K_1$obtained from graphical presentation with four generators as in Figure 5.

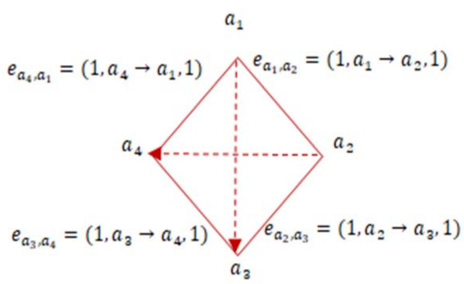

Figure (5). The connected 2-complex graph 4K_1

while ${}^4K_2$ is

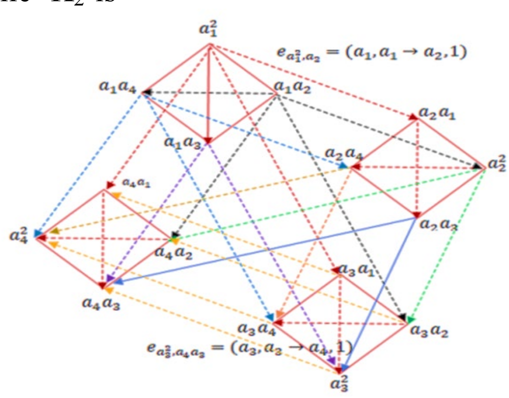

Figure (6). The connected 2-complex graph ${}^4K_2$

Note that ${}^4K_2$is four copies of ${}^4K_1$ and each vertex in each copy are joined together respectively. Likewise with four copies of ${}^4K_2$, the square complex graph ${}^4K_3$ may be obtained by repeating similar procedures with the result ${}^4K_4$ and so on. From those diagrams we can conclude some properties.

LEMMA 3.1 Let ${}^4S =< a_1, a_2, a_3, a_4: a_i = a_j; 1 \leq i < j \leq 4 >$ be a graphical presentation. A connected square complex graph ${}^4K_n$ contains 4^n vertices.

PROOF: By induction, for $n = 1$ the number of all vertices in ${}^4K_1$ is 4. Assume the number of all vertices in ${}^4K_k$ is 4^k where $n = k$. Now we will prove that the number of all vertices in ${}^4K_{k+1}$ is 4^{k+1}. By the definition of ${}^4K_{k+1}$ is four copies of ${}^4K_k$ and assumption, then the number of all vertices in ${}^4K_{k+1}$ is $4 \cdot 4^k = 4^{k+1}$.

LEMMA 3.2

Let ${}^4S =< a_1, a_2, a_3, a_4: a_i = a_j; 1 \leq i < j \leq 4 >$ be a graphical presentation. A connected square complex graph ${}^4K_{n+1}$ is four copies of ${}^4K_n$. Thus, if there is e_n edges in ${}^4K_n$ then the number of edges in ${}^4K_{n+1}$is $4e_n$ plus all edges between squares in ${}^4K_{n+1}$, which is $e_{n+1} = 4e_n + 4^n$.

LEMMA 3.3 Vertices U and V are connected if and only if $L(U) = L(V)$.

LEMMA 3.4 If $L(U) = L(V)$ then $\pi_1(K^4(S), U) = \pi_1(K^4(S), V)$

LEMMA 3.5 Vertices of ${}^4K_n$ are all words of length n.

The following theorem addresses the issue of covering graph.

MAIN THEOREM 1: A connected square complex graph ${}^4K_{i+1}$is the covering space for ${}^4K_i$ for all $i \in N$.

PROOF: Our claim is to prove ${}^4K_{i+1}$is the covering space of ${}^4K_i$, for all $i \in N$. We will confirm by induction.
for $i = 1$. The aim is to confirm that ${}^4K_2$ is the covering graph for ${}^4K_1$.

It is noticed that the square complex graphs ${}^4K_1$ and ${}^4K_2$ are connected since that for any two vertices taken in these square complex graphs, there will be a path connecting them. So, let $\rho : {}^4K_2 \to {}^4K_1$ defined by:

$\rho(a_1^2) = \rho(a_2a_1) = \rho(a_3a_1) = \rho(a_4a_1) = a_1$,
$\rho(a_1a_2) = \rho(a_2^2) = \rho(a_3a_1) = \rho(a_4a_1) = a_2$,
$\rho(a_1a_3) = \rho(a_2a_3) = \rho(a_3^2) = \rho(a_4a_3) = a_3$,
$\rho(a_1a_4) = \rho(a_2a_4) = \rho(a_3a_4) = \rho(a_4^2) = a_4$,

$\rho(e_{a_1^2,a_1a_2}) = \rho(e_{a_1^2,a_2a_1}) = \rho(e_{a_1^2,a_3a_1}) = \rho(e_{a_1^2,a_4a_1}) = e_{a_1,a_2}$,
$\rho(e_{a_1a_2,a_1a_3}) = \rho(e_{a_2^2,a_2a_3}) = \rho(e_{a_3a_2,a_3^2}) = \rho(e_{a_4a_2,a_4a_3}) = e_{a_2,a_3}$,
$\rho(e_{a_1a_3,a_1a_4}) = \rho(e_{a_2a_3,a_2a_4}) = \rho(e_{a_3^2,a_3a_4}) = \rho(e_{a_4a_3,a_4^2}) = e_{a_3,a_4}$,
$\rho(e_{a_1a_4,a_1^2}) = \rho(e_{a_2a_4,a_2a_1}) = \rho(e_{a_3a_4,a_3a_1}) = \rho(e_{a_4^2,a_4a_1}) = e_{a_4,a_1}$
$\rho(e_{a_1^2,a_2a_1}) = 1_a, \rho(e_{a_1^2,a_3a_1}) = 1_a, \rho(e_{a_1^2,a_3a_1}) = 1_a, \rho(e_{a_1^2,a_4a_1}) = 1_a$,
$\rho(e_{a_1^2,a_1a_3}) = 1_a$,

In order to prove that ${}^4K_{i+1}$ is the covering graph of ${}^4K_i$, it must be proved that ρ is a locally bijective for all vertices. It is found that

$$\rho_{star(a_1^2)}: \{e_{a_1^2,a_1a_2}, e_{a_1^2,a_1a_3}, e_{a_1^2,a_1a_4}, e_{a_1^2,a_2a_1}, e_{a_1^2,a_3a_1}, e_{a_1^2,a_4a_1}\} \to \{e_{a_1,a_2}, 1_{a_1}, 1_{a_1}, 1_{a_1}, 1_{a_1}, 1_{a_1}\}$$
$$\rho_{star(a_1a_2)}: \{e_{a_1a_2,a_1a_3}, e_{a_1a_2,a_1a_4}, e_{a_1a_2,a_2^2}, e_{a_1a_2,a_3a_2}, e_{a_1a_2,a_4a_2}, e_{a_1a_2,a_4a_1a_2}\} \to \{e_{a_2,a_3}, 1_{a_1a_2}, 1_{a_1a_2}, 1_{a_1a_2}, 1_{a_1a_2}, 1_{a_1a_2}\}$$

It can be seen that in this case ρ is not a locally bijective. This part gets even more complicated. So, ρ is redefined to prove it is a locally bijective. Indeed, there are two cases to define the above ρ; the first case is $\rho(e_{Wa,Wb}) = e_{a,b}$, while the second case is $\rho(e_{aW,bW}) = 1_W$, and hence can be ignored (refer to **Definition 13**).

$\rho_{star(a_1^2)}: \{e_{a_1^2,a_1a_2}\} \to \{e_{a_1,a_2}\}$

$\rho_{star(a_1a_2)}: \{e_{a_1a_2,a_1a_3}\} \to \{e_{a_2,a_3}\}$

So, when ρ is a locally injective and locally surjective, consequently ρ is a locally bijective. Therefore ${}^4K_2$ is the covering complex for ${}^4K_1$. This finding, in turn, bears evidence that the previous theorem (i.e., ${}^4K_{i+1}$ is the covering graph of ${}^4K_i$ for $i \in N$) is true.
For $i = k - 1$, assume ${}^4K_k$ is the covering space for ${}^4K_{k-1}$.

Now for $i = k$, the aim here is to confirm that ${}^4K_{k+1}$ is the covering complex for ${}^4K_k$. It is noticed that the square complex graphs ${}^4K_k$ and ${}^4K_{k+1}$ are connected, since that for any two vertices taken in these square complex graphs, there will be a path connecting them. So it remains here to prove that ${}^4K_{k+1}$ is the covering complex for ${}^4K_k$. To attest that, let $\rho : {}^4K_{k+1} \to {}^4K_k$ defined by $\rho(Wa) = a$, where W is a word on $\{a_1, a_2, a_3, x_4\}$ of length k, $a, b \in \{a_1, a_2, a_3, a_4\}$, $\rho(e_{Wa,Wb}) = e_{a,b}$, and $\rho(e_{aW,bW}) = e_{a,b}$.

To prove ρ is a locally bijective, it must be proved that ρ is locally bijective for all vertices. Here, the same procedure taken previously for $i = 1$ will be repeated. So, $\rho_{x_i^{k+1}} : star(a_i^{k+1}) \to star(a_i^k), i = \{1,2,3,4\}$ is a locally injective and locally surjective; thus, ρ is a locally bijective. This finding, in turn, bears evidence that the previous theorem (i.e. ${}^4K_{k+1}$ is the covering complex for ${}^4K_k$) is true.

The following theorem highlights the technique of creating normal subgroup of one generator and discusses how to cover complex ${}^4K_{H_1}$ for the connected square complex ${}^4K_1$.

MAIN THEOREM 2: Consider a connected square complex graph ${}^4K_1$ as shown in Figure 1, such that $G = \pi_1({}^4K_1, a_1)$ contains γ_1, where $\gamma_1 =< e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}e_{a_4,a_1} >$. If H_1 is the smallest normal subgroup of G containing $< \gamma_1^2 >$, then the covering space ${}^4K_{H_1}$ is an octagonal shape.

PROOF: Use the notion of $H(\beta_1) = H(\beta_2) \Leftrightarrow [\beta_1\beta_2{}^{-1}] \in H$. From ${}^4K_1$, $\pi_1({}^4K_1)$ can be obtained. Fix a vertex a_1 in ${}^4K_1$. Now, for any normal subgroup of $\pi_1({}^4K_1, a_1)$, there exists a unique covering space. Start by choosing basic $H[\alpha]$ where α is a path such that $i(\alpha) = a_1$, $\tau(\alpha) = v$ for every vertex v in ${}^4K_1$. As a result, these basic $H[1]$, $H[e_{a_1,a_2}]$, and $H[e_{a_1,a_2}e_{a_2,a_3}]$ will be selected, and then all possible edges can be determined, as shown in Table 1.

Table (1). Edges from H[1] in ${}^4K_{H_1}$

Edges	Initial	Terminal
$(H[1], e_{a_1,a_2})$	$H[1]$	$H[e_{a_1,a_2}]$
$(H[1], e_{a_1,a_4})$	$H[1]$	$H[e_{a_1,a_4}]$

Since $\rho_H[H[1]] = a_1$ and $star(a_1) = 2$, then $star(H[1]) = 2$. Consider a vertex a_1; the vertex in ${}^4K_{H_1}$ is $H[1]$, and $H[1]$ in ${}^4K_{H_1}$ maps to a_1. From $a_1 \to a_2$ in ${}^4K_1$, the vertex in ${}^4K_{H_{21}}$ is $H[e_{a_1,a_2}]$, and the edge is $(H[1], e_{a_1,a_2})$. $H[e_{a_1,a_2}]$ in ${}^4K_{H_1}$ maps to a_2 in ${}^4K_1$, as shown in Figure 7.

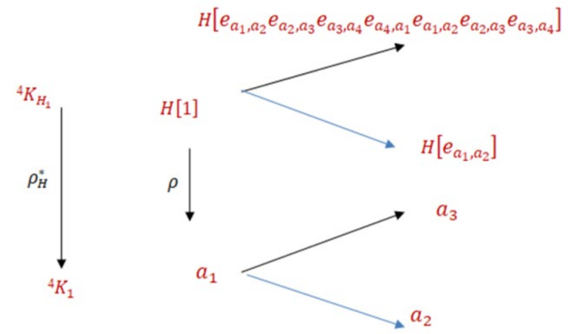

Figure (7). Mapping from ${}^4K_{H_1}$ to ${}^4K_1$

From a_1 to a_2 to a_3, the vertex in ${}^4K_{H_1}$ is $H[e_{a_1,a_2}e_{a_2,a_3}]$ and the edge is $(H[e_{a_1,a_2}], e_{a_1,a_2}e_{a_2,a_3})$. The vertex in ${}^4K_{H_1}$ maps to a_3 in ${}^4K_1$. Now from a_1 to a_2 to a_3 to a_4, the vertex in ${}^4K_{H_{21}}$ is $H[e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}]$ and the edge is $(H[e_{a_1,a_2}e_{a_2,a_3}], e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4})$. The vertex in ${}^4K_{H_1}$ maps to a_4 in ${}^4K_1$. Again from a_1 to a_2 to a_3 to a_4 to a_1 in ${}^4K_1$, the vertex in ${}^4K_{H_1}$ is $H[e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}]$ and the edge is $(H[e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}], e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}e_{a_4,a_1})$. This vertex in ${}^4K_{H_1}$ maps to a_1 in ${}^4K_1$. Since H_1 is the smallest normal subgroup of G containing $< \gamma_1^2 >$, a duplicate is needed to get all the vertices. So, the right cosets are as the following:

- $H[1]$
- $H[e_{a_1,a_2}]$
- $H[e_{a_1,a_2}e_{a_2,a_3}]$
- $H[e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}]$
- $H[e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}e_{a_4,a_1}]$
- $H[e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}e_{a_4,a_1}e_{a_1,a_2}]$
- $H[e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}e_{a_4,a_1}e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}]$.

The edges are:

- $(H[1], e_{a_1,a_2})$
- $(H[e_{a_1,a_2}], e_{a_1,a_2}e_{a_2,a_3})$
- $(H[e_{a_1,a_2}e_{a_2,a_3}], e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4})$
- $(H[e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}], e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}e_{a_4,a_1})$
- $(H[e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}e_{a_4,a_1}], e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}e_{a_4,a_1}e_{a_1,a_2})$
- $(H[e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}e_{a_4,a_1}e_{a_1,a_2}], e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}e_{a_4,a_1}e_{a_1,a_2}e_{a_2,a_3})$
- $(H[e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}e_{a_4,a_1}e_{a_1,a_2}e_{a_2,a_3}], e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}e_{a_4,a_1}e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4})$
- $(H[1], e_{a_2,a_3}e_{a_3,a_4}e_{a_4,a_1}e_{a_1,a_2}e_{a_2,a_3})$.

So, ${}^4K_{H_1}$ will look like the depicted form in Figure 8, where

- $e_1 = e_{a_2,a_3}$
- $e_2 = e_{a_2,a_3}e_{a_3,a_4}$
- $e_3 = e_{a_2,a_3}e_{a_3,a_4}e_{a_4,a_1}$
- $e_4 = e_{a_2,a_3}e_{a_3,a_4}e_{a_4,a_1}e_{a_1,a_2}$
- $e_5 = e_{a_2,a_3}e_{a_3,a_4}e_{a_4,a_1}e_{a_1,a_2}e_{a_2,a_3}$
- $e_6 = e_{a_2,a_3}e_{a_3,a_4}e_{a_4,a_1}e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}$.

Now let $\rho_H^* : {}^4K_{H_1} \to {}^4K_1$ defined by $\rho_H^*(H[1]) = a_1, \rho_H(H[e_{a,b}]) = a, b$, $\rho_H(H[\alpha], e_{a,b}) = e_{a,b}$. This map can be viewed as locally bijective, so the covering complex ${}^4K_{H_1}$ will be an octagonal shape.

Since a_1 is a vertex of the connected square complex ${}^4K_1$ and $H[1]$ lies over a_1, then $\rho_H^*: \pi_1({}^4K_{H_1}, H[1]) \to \pi_1({}^4K_1, a_1)$ is injective. Upon that, it is found that $\rho_H^*: \pi_1({}^4K_{H_1}, H[1]) \to \mathrm{Im}\, \rho_H^* = H$. As a result, $H = \pi_1({}^4K_{H_1}, H[1])$ can be viably considered as a subgroup of $G = \pi_1({}^4K_1, a_1)$.

Now, we determine the generators for the fundamental group $\pi_1({}^4K_{H_1}, H[1])$ by using maximal subtree technique, choose a maximal subtree $T({}^4K_{H_1})$ for ${}^4K_{H_1}$ and suppose that $T({}^4K_{H_1})$ starts from $H[1]$.

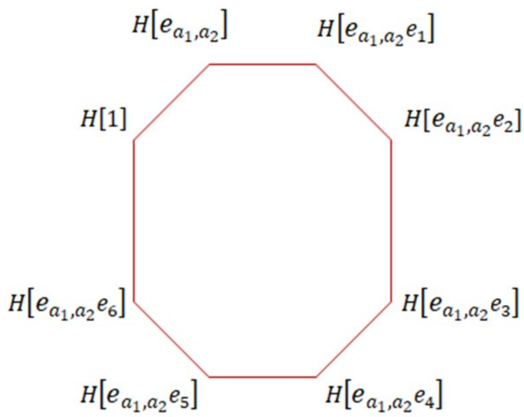

Figure (8). Covering complex ${}^4K_{H_1}$

So

$T(K_{H_{1_2}}) = (H[1], e_{a_1,a_2})(H[e_{a_1,a_2}], e_{a_1,a_2}e_{a_2,a_3})(H[e_{a_1,a_2}e_{a_2,a_3}], e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4})$ $(H[e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}], e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}e_{a_4,a_1})(H[e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}e_{a_4,a_1}], e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}e_{a_4,a_1}e_{a_1,a_2})(H[1], e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}e_{a_4,a_1}e_{a_1,a_2}e_{a_2,a_3})$. Hence, the edge $(H[1], e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}e_{a_4,a_1}e_{a_1,a_2}e_{a_2,a_3}e_{a_3,a_4}) \notin T({}^4K_{H_1})$. Consequently, $T({}^4K_{H_1})$ will look like the depicted form in Figure 9.

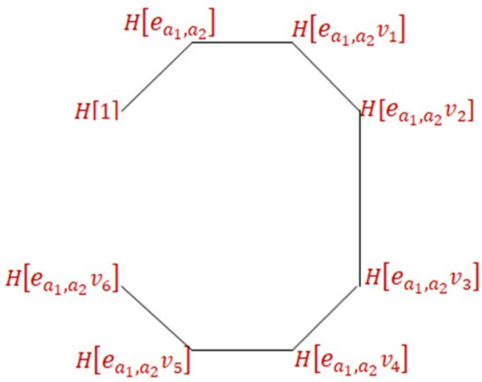

Figure (9). Maximal subtree $T({}^4K_{H_1})$

CONCLUSION

The paper introduced how to construct the connected 2-complex graph ${}^4K_i, i \in N$ of a diagram group from semigroup presentations ${}^4S =< a_1, a_2, a_3, a_4 : a_i = a_j ; 1 \le i < j \le 4 >$. The paper also showed that the square complex graphs were connected according to the length of words, and it proved that the connected square complex graph ${}^4K_{i+1}$ is the covering graph for ${}^4K_i$, for all $i \in N$. The article discussed how to determine the covering space ${}^4K_{H_i}$ for all connected square complex graph ${}^4K_i, i \in N$ by selecting normal subgroups from the diagram group. This paper also presented some diagrams.

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