



Totally Volume Integral of Fluxes for Discontinuous Galerkin Method (TVI-DG) I- Unsteady Scalar One Dimensional Conservation Laws

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Abstract: The volume integral of Riemann flux in the discontinuous Galerkin (DG) method is introduced in this paper. The boundaries integrals of the fluxes (Riemann flux) are transformed into volume integral. The new family of DG method is accomplished by applying divergence theorem to the boundaries integrals of the flux. Therefore, the (DG) method is independent of the boundaries integrals of fluxes (Riemann flux) at the cell (element) boundaries as in classical (DG) methods. The modified streamline upwind Petrov-Galerkin method is used to capture the oscillation of unphysical flow for shocked flow problems. The numerical results of applying totally volume integral discontinuous Galerkin method (TVI-DG) are presented to unsteady scalar hyperbolic equations (linear convection equation, inviscid Burger's equation and Buckley-Leverett equation) for one dimensional case. The numerical finding of this scheme is very accurate as compared with other high order schemes as the weighted compact finite difference method WCOMP.

Key words: Scalar conservation laws; Higher order methods, Discontinuous Galerkin; Divergence theorem

INTRODUCTION

There has been a surge of researches activities in high order methods as spectral volume (SV) method, spectral difference (SD) method, the weighted essentially non-oscillatory (WENO) method and (DG) method. Most of the aforementioned methods have a common feature: they achieve high order accuracy by locally approximating the state variables (numerical solutions) as high order polynomials inside the element. While WENO method achieves a high order accuracy by approximating the state variables as high order polynomials over a stencil (groups of cells or elements). In this work, we concerned ourselves to the compact and weighted scheme which is the DG method.

The DG method is introduced by Reed and (Hill in 1973) for neutron transport problems and

then developed for fluid dynamics by Cockburn and Shu in series of papers among them (Cockburn *et al.*, 1989, Cockburn and Shu 1989, Cockburn *et al.*, 1990, Cockburn 2001). (Huynh 2007) introduced a flux reconstruction (FR) approach, in which the formulation is capable of unifying several popular methods including the discontinuous Galerkin method, staggered-grid method, spectral difference method and spectral volume method into a single formula. The final mathematical form of the discretized governing equation is governing equation in the differential form. After that, (Wang and Gao 2009) extended (FR) approach to multidimensional flow and unstructured mesh under the named lifting collocation penalty (LCP) formulation. Therefore, the differences between DG and other methods lie in the definition of degrees of freedom (DOFs)

and how the DOFs are updated (Wang and Gao 2009).

It is well known that the discontinuous Galerkin method is as an efficient and low error magnitude than the other methods. In the DG formulation, the boundary flux is integrated over the boundary of the cell or element as traditional methods like finite volume (FV) methods. While for its development the weighted function at the boundary of the cell can be transformed into the correction function $g(\xi)$ for (FR) or lifting coefficients $\alpha_{i,j}$ for LCP formulations. Thus $g(\xi)$ and $\alpha_{i,j}$ are dependent on the weighted functions over the boundaries (Gao and Wang 2009, Wang and Gao 2009). Therefore the weighted functions at the boundary play an important rule for boundary flux calculation in the DG method and its development. In general, there are two types of flux integrals, the first one is the volume integral of the physical flux over the entire element domain, and the second type of integral is the boundary integral of the Riemann flux (Godunov flux) over the boundaries of the elements (over the surface areas of the element).

This difficulty motivated us to introduce a new family of DG methods independent of the weighting functions at the boundaries. Therefore no boundary integral is needed for this new formulation.

The paper is organized as follows: Section 2 introduced the new DG method formulations. The verification of the new formulation is introduced in section 3. Finally, conclusion remarked is introduced into section 4.

Totally Volume Integral (TVI) DG Method Formulation

Space Discretization

For the convenience of discussion, a review for DG semi-discretization for partial differential equations (PDE) is introduced. This can be done by firstly considering the conservation laws in divergence form:

$$Q_t + \nabla \cdot \mathbf{F} = 0. \tag{1}$$

The numerical solution of Eqn. (1) is sought on the computational domain Ω subject to proper initial and boundary condition. where Q is the conservative variable and \mathbf{F} is the conservative flux vector.

In Eqn. (1) Q and \mathbf{F} are scalar or column, representing scalar or system of equations. The weighted residual formulation is obtained by multiplying Eqn. (1) by a scalar test function (weighting function) W and integrating by parts over the domain Ω

$$\int_{\Omega} [WQ_t - \nabla W \cdot \mathbf{F}(Q)] d\Omega + \int_{\partial\Gamma} W\mathbf{F}(Q) \cdot \mathbf{n} d\Gamma = 0. \tag{2}$$

A discretization analogue of Eqn. (2) over each element can be obtained by subdividing the computational domain Ω into N non-overlapping elements $\Omega = \bigcup_{k=1}^N \Omega_h$. By applying Eqn. (2) to each element Ω_h , the semi-discrete analogue of Eqn. (2.2) over the computational grid yields:

$$\int_{\Omega_h} \left[W_h \frac{\partial Q_h}{\partial t} - \nabla W_h \cdot \mathbf{F}(Q_h) \right] d\Omega_h + \int_{\Gamma_h} W_h \mathbf{F}(Q_h) \cdot \mathbf{n} d\Gamma_h = 0, \tag{3}$$

Γ_h denotes the boundary of the element Ω_h and \mathbf{n} is outward vector normal to the boundary. Let Q_h and W_h represent the finite element approximation to the analytical solution Q and the test function W respectively where Q_h and W_h are approximated by a piecewise polynomial function of degree k , which is continuous within each element and discontinuous between the elements interfaces.

$$Q_h(x, t) = \sum_{j=1}^{j=n} \varphi_j Q_j(t) \text{ and } W_h(x) = \sum_{j=1}^{j=n} \varphi_j W_j, \tag{4}$$

Where n is the dimension of the polynomial space p^k and φ_j is the basis of the polynomial. The expansion coefficients $Q_j(t)$ and W_j denotes the degrees of freedom (DOFs) of the numerical solution and the test function in element Ω_k , respectively. Thus the summation in eqn. (3) is equivalent to the following system of n equations

$$\int_{\Omega_h} [\varphi_j Q_t - \nabla \varphi_j \cdot F(Q_h)] d\Omega_h + \int_{\Gamma_h} \varphi_j F(Q_h) \cdot \mathbf{n} d\Gamma_h = 0, \quad 1 \leq j \leq n, \quad (5)$$

since the discontinuities are permitted at the interfaces of elements in the DG method. Because the approximated solution is discontinuous at the element boundaries, the interface flux is not uniquely defined. In this stage, the Riemann fluxes used in the Godunov finite volume method are borrowed.

The normal flux function $F(Q_h) \cdot \mathbf{n}$ appearing in the last terms of eqn(5) is replaced by a numerical Riemann flux function $F_{up} = F(Q_L, Q_R, \mathbf{n})$ that depends on Q_L and Q_R which are the approximated solutions of the conservative state variables Q_h at the left and right side of the element boundary, respectively. In order to guarantee consistency and conservation, the Riemann flux must satisfy the following

$$F_{up} = F(Q_L, Q_R, \mathbf{n}) = F(Q_h) \cdot \mathbf{n} \text{ and } -F_{up} = F(Q_L, Q_R, -\mathbf{n}) = -F(Q_h) \cdot \mathbf{n}. \quad (6)$$

In the present work, the Riemann flux is approximated by using Lax and Friedrich (LF) flux for nonlinear flux. This scheme is called discontinuous Galerkin method of degree k as given in the classical form, or in short notation DG (k) method. The surface and volume integrals in Eqn. (5) are calculated in case of DG method by using $2k$ and $2k+1$ order accurate Gauss quadrature formulas, respectively.

In order to unify the integrals (surface integral and volume integral), the totally volume integral of the upwind flux scheme for DG method is used for this purpose. The relation between the surface and volume integrals for any vector A is given by the divergence theorem as

$$\oint_{\Gamma} A \cdot \mathbf{n} d\Gamma = \iiint_V \nabla \cdot A dV, \quad (7)$$

where Γ and V are surface and volume of the problem domain. The totally volume integral DG method is accomplished by applying the divergence theorem to the last term of Eqn. (5) and rearranging to give the following form

$$\int_{\Omega_h} \left[\varphi_j \frac{\partial Q_h}{\partial t} - \nabla \varphi_j F(Q_h) + \nabla \varphi_j F_{up} + \varphi_j \nabla \cdot F_{up} \right] d\Omega_h = 0, \quad (8.a)$$

for one dimensional case Eqn. (8.a) can be written as:

$$\int_{\Omega_h} \left[\varphi_j \frac{\partial Q_h}{\partial t} - \frac{\partial \varphi_j}{\partial x} F(Q_h) + \frac{\partial \varphi_j}{\partial x} F_{up} + \varphi_j \frac{\partial F_{up}}{\partial x} \right] d\Omega_h = 0. \quad (8.b)$$

The Riemann or upwind flux vectors are approximated by polynomial of order k as done for the state variable in Eqn. (4). $F(Q_h) = \sum_{i=1}^{i=n} \phi_i F_i(Q_h)$ $F_{up} = \sum_{i=1}^{i=n} \phi_i F_{up,i}$, the last two terms of Eqns. (8.b) can be companied into one term as follows

$$\int_{\Omega_h} \left[\varphi_j \frac{\partial Q_h}{\partial t} - \frac{\partial \varphi_j}{\partial x} \varphi_i F_i(Q_h) + \left(\frac{\partial \varphi_j}{\partial x} \varphi_i + \varphi_j \frac{\partial \varphi_i}{\partial x} \right) F_{up,i} \right] d\Omega_h = 0. \quad (9)$$

Equation (9) is the DG method in totally volume integral form.

Coordinate Transformation

In order to achieve an efficient implementation, all elements are transformed from the computational space (x,y,z) into standard space (ζ, η, ξ) . Consequently, all partial derivatives with respect to the standard space are related to the partial derivative in the computational space as in the finite element methods. For one dimensional case, the value of the x can be obtained as:

$$x = \sum_{i=1}^{i=N} x_j \varphi_j(\zeta). \quad (10)$$

The derivative of x with respect to ζ is obtained as:

$$\frac{\partial x}{\partial \zeta} = x_\zeta = \sum_{i=1}^{i=N} x_j \frac{\partial \varphi_j(\zeta)}{\partial \zeta}. \quad (11)$$

The derivatives of any function with respect to the standard coordinate can be written as:

$$\frac{\partial(\)}{\partial \zeta} = \frac{\partial(\)}{\partial x} \frac{\partial x}{\partial \zeta} = \frac{\partial(\)}{\partial x} x_{\zeta}, \text{with } |J| = |x_{\zeta}|. \quad (12)$$

Where $|J|$ is the determinant of Jacobian matrix. Also, the derivatives of any function with respect to physical coordinates can be written as:

$$\frac{\partial(\)}{\partial x} = \frac{\partial(\)}{\partial \zeta} \zeta_x, \text{with } |J^{-1}| = |\zeta_x|. \quad (13)$$

From equations (11) and (13), $x_{\zeta} = 1/\zeta_x$, with $n_{\zeta} = \zeta_x/|\zeta_x| = 1$. Thus no negative values of the Riemann flux $F_{up} = F(Q_L, Q_R, \mathbf{n}_{\square})$ at the boundaries. By substituting into Eqn.(9) and rearrangement yields:

$$\int_{\Omega_h} \left[\varphi_j \frac{\partial Q_h}{\partial t} - \frac{\partial \varphi_j}{\partial \zeta} \varphi_i (\zeta_x F_i(Q_h)) + \left(\frac{\partial \varphi_j}{\partial \zeta} \varphi_i + \varphi_j \frac{\partial \varphi_i}{\partial x} \right) (\zeta_x F_{up,i}) \right] d\Omega_h = 0. \quad (14)$$

Finally after the spatial discretization is accomplished, equation (14) can be written into the following form

$$M \frac{dQ}{dt} = R(Q), \quad (15)$$

where $R(Q)$ is the residual and M is called the consistent mass matrix

Time Integral

The semi-discrete equation as Eqn. (15) can be integrated in time using explicit methods. The explicit three-stage third-order TVD Runge-Kutta scheme RK(3,3) and five-stage fourth order RK(5,4) are the widely used methods given in many references among them (Gao and Wang 2009). The RK(3,3) can be expressed in the following form:

$$Q^{(1)} = Q^n + \Delta t M^{-1} R(Q^n) \quad (16.a)$$

$$Q^{(2)} = 3/4 Q^{(1)} + 1/4 [Q^n + \Delta t M^{-1} R(Q^{(1)})] \quad (16.b)$$

$$Q^{n+1} = 1/3 Q^n + 2/3 [Q^{(2)} + \Delta t M^{-1} R(Q^{(2)})] \quad (16.c)$$

This method is linearly stable for a Courant number less than or equal to 1.

Numerical Results

All of the computations are performed on a Compaq laptop computer (2.33 GHz Intel (R) Core (TM) 2 CPU T7600 with 4G Bytes memory) using Ubuntu 14.05 Linux operating system. The code was written in C Language and compiled with the default gcc compiler. As a preliminary test we apply the totally volume integral discontinuous Galerkin method to several one-dimensional examples involving linear advection equations, inviscid Burger's equation and Buckley-Leverett equations.

The global error is calculated as the difference between the exact solutions and the numerical solutions. The discretize L_1 norm error is given as

$$L_1 = \sum_{j=1}^{j=N} \sum_{i=1}^{i=edof} |Q^{ex} - Q_i^h| / t_{dof},$$

where N is the total number of elements, $edof$ is the element degree of freedom and $t_{dof} = (N \times edof)$ is the total degree of freedom.

Numerical Tests and Comparison.

Example-1

The first example is linear advection equation considered in many references among them (Zhang *et al.*, 2008).

$$\frac{\partial Q}{\partial t} + \frac{\partial F(Q)}{\partial x} = 0, \quad (1)$$

with $F(Q) = Q^2$. The initial condition is given as $Q(x,0) = \sin^4(\pi x)$ with periodical boundary conditions. The exact solution is $Q(x,t) = \sin^4(\pi(x-t))$. The domain $[-1,1]$ is divided into N equally space elements. The approximated solutions are constructed from polynomials of orders k from 1 to 3. The RK (3,3) is used for $k = 1$ and 2, while RK (5,4) is used in case of $k = 3$, where the RK methods are used for evaluating the time integral part. The numerical results are obtained at time $t = 1.0$. Table (1) exhibits the L_1 error and order of accuracy using TVI-DG method. Whereas Table (2) reveals the L_1 error and order of accuracy using weighted compact

method of forth and sixth order of accuracy from (Zhang *et al.*, 2008). Figure(1) displays the numerical solution at $t = 1.0$ using polynomial of order $k = 2$. Figure 2 reveals L_1 error of TVI-DG method with polynomials from 1 to 3 and the L_1

error of the weighted compact method of orders 4 and 6 from (Zhang *et al.*, 2008). Figure (2) and Table (2) show that TVI-DG method has lower error magnitude as compared with weighted compact method.

Table (1) L_1 error and the order of accuracy for 1D linear advection eqn. with periodic boundary conditions at $t = 1$ by using TVI-DG method with polynomials of orders $k = 1$ to 3.

N	$k=1$		$k=2$		$k=3$	
	L_1 error	Order	L_1 error	order	L_1 error	order
10	2.90933e-01	-	6.005329e-02	-	3.991401e-03	-
20	9.40716e-02	1.628	4.146367e-03	3.856	1.021990e-04	5.287
40	2.811773e-02	1.742	2.284089e-04	4.182	5.570254e-06	4.197
80	4.257114e-03	2.723	2.093349e-05	3.447	3.388487e-07	4.039
160	5.529017e-04	2.944	2.419287e-06	3.113	2.12720e-08	3.993

Table (2) L_1 error and the order of accuracy for 1D linear advection eqn. from (Zhang *et al.*, 2008)

Method	N	L_1 error	L_1 order
WCOMP4	10	3.56e-1	-
	20	1.42e-1	1.33
	40	2.62e-2	2.44
	80	2.21e-3	3.57
	160	1.64e-4	3.76
WCOMP6	10	3.56e-1	-
	20	9.27e-2	1.94
	40	7.22e-3	3.68
	80	3.06e-4	4.56
	160	1.10e-6	8.12

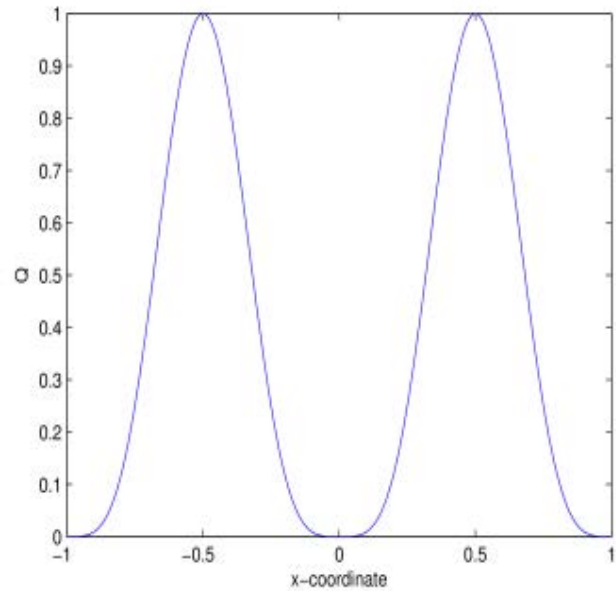


Figure (1). The numerical solution of 1D linear advection eqn. with initial condition $Q = \sin^4(\pi x)$ at $t = 1.0$, $N = 200$ elements by using polynomial of order $k = 2$

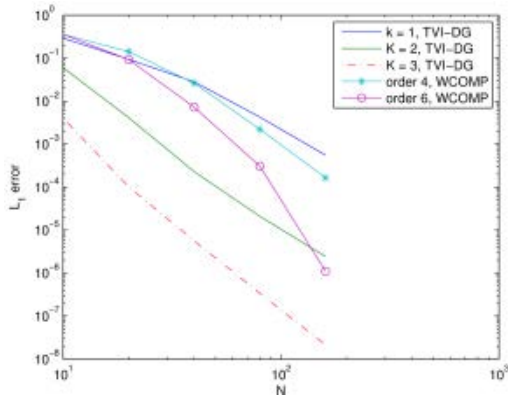


Figure (2). L_1 errors for TVI-DG, with $k=1$ to 3 and weighted compact method from (Zhang *et al.*, 2008).

Example-2

The second example is the linear advection equation, eqn. (1) with $F(Q) = Q$. The initial condition $Q(0,x) = 0.0$ and the boundary condition given as $Q(0,t) = \sin\left(\frac{1}{2}\pi t\right)$. The problem is considered in many references among them Ref. (Liu *et al.*, 2010). The problem domain $x \in [0,100]$ is divided into 200 elements. The approximated solutions are constructed from polynomials of order k from 2 to 4. Due to the smooth solution; there is no need for using the stabilization technique. The time part is evaluated using RK(3,3) and RK(5,4). Figures 3 to 5 display the numerical solutions at $t = 20, 40$ and 60 , respectively. The figures demonstrate that the TVI-DG is a very efficient method for solving problems with sine wave propagation from the boundary to the main domain, without losses in the wave amplitude in case of long time intervals.

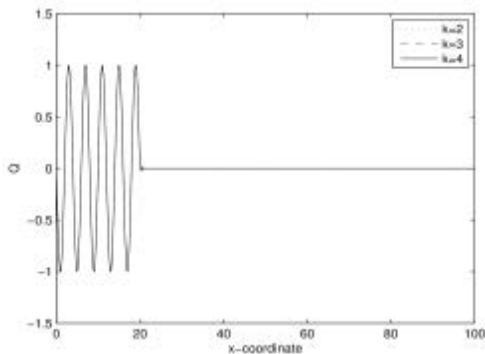


Figure (3). The numerical solution of example 2. by using TVI-DG with $k=2$ to 4 at time $t=20$.

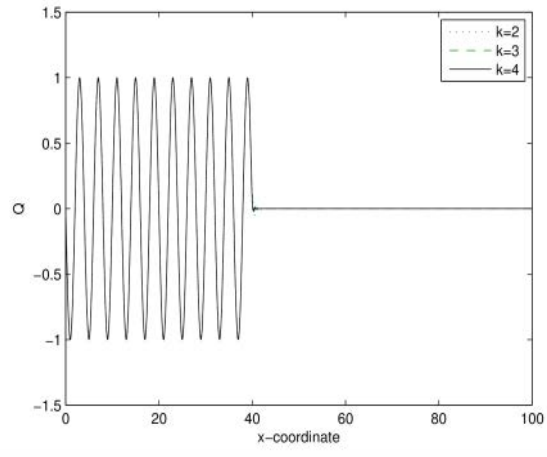


Figure (4). The numerical solution of example 2. by using TVI-DG with $k=2$ to 4 at time $t=40$.

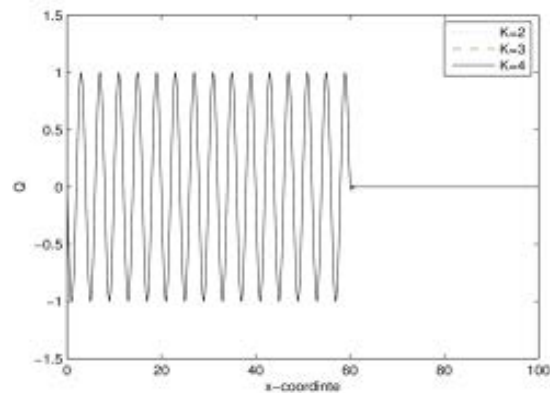


Figure (5). The numerical solution of example 2. by using TVI-DG with $k=2$ to 4 at time $t=60$.

Example-3

The third test example is the inviscid Burger's equation, considered in many references among them (Wang *et al.*, 2008).

$$\frac{\partial Q}{\partial t} + \frac{\partial F(Q)}{\partial x} = 0, \text{ with } F(Q) = \frac{1}{2} Q^2. \quad (2)$$

The initial condition is $u(0, x) = \sin(\pi x)$ with periodical boundary conditions. The problem domain $[0,2]$ is divided into 150 equally spaced elements. The approximated solutions are constructed from polynomials of orders k from 2 to 4. The RK (5,4) are used for evaluating the time integral part. The numerical results are obtained at time $t = 1$. Due to discontinuity, the modified streamline - upwind stabilization technique is used to capture the unphysical

oscillation in the flow problem. Figure (6). display the numerical solutions using TVI-DG method at $t = 1$. The figure reveals that the TVI-DG method is very efficient and there is no significant unphysical flow in case of shock wave problems.

Example-4

The fourth example is Buckley-Leverett equation for two phase flow(Xin and Flaherty 2006).

$$\frac{\partial Q}{\partial t} + \frac{\partial F(Q)}{\partial x} = 0, \text{with } F(Q) = \frac{Q^2}{Q^2 + \frac{1}{2}(1-Q)^2}. \quad (3)$$

The initial condition is given as $Q(x, 0) = 1, x < 0$
 $0, x \geq 0$. The problem domain $[-1, 2.5]$ is divided into 200 equal elements. The approximated solutions are constructed from polynomials of orders K from 2 to 4. The RK (5,4) are used for evaluating the time part. The numerical solution is obtained at time $t = 1.0$. Figure (7) demonstrates that the TVI-DG method with modified streamline upwind stabilization technique is very efficient and there is no nonphysical oscillation of the numerical solution for the shocked flow where the solution involves one moving shock wave followed by expansion wave(Xin and Flaherty 2006). However there is no closed form of the equation (exact solution), thus the exact solution can be obtained by using 1000 element.

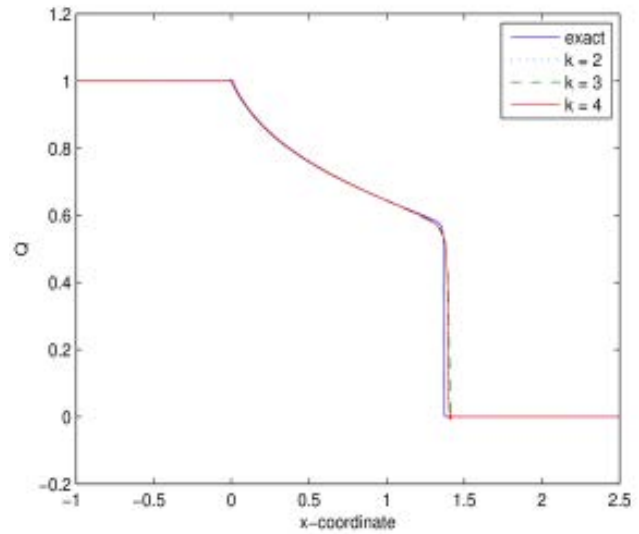


Figure (7). The numerical solution of example 4. by using TVI-DG with $k = 2$ to 4 at time $t = 1$.

Example-5

The fifth test example is the Buckley-Leverett equation with the standard parameters, considered in (Xin and Flaherty 2006). The standard parameter of the flux is $F(Q) = \frac{Q^2}{Q^2 + \frac{1}{4}(1-Q)^2}$. The initial condition is given as:

$$Q(x, 0) = \begin{cases} 1, & -0.5 \leq x \leq 0 \\ 0, & x \geq 0 \end{cases}$$

The problem domain $[-1, 1]$ is divided into 200 equal elements. The approximated solutions are constructed from polynomials of orders $k = 2$ and 3. The RK (5,4) are used for evaluating the time integral part. The numerical results are obtained at time $t = 0.4$.

Figure (8) displays the numerical solutions at time $t = 0.4$, by using TVI-DG method with $k = 2$ and 3 and the stabilization technique is used to capture the unphysical oscillation in the flow. The solution involves two moving shock waves each followed by an expansion wave (Xin and Flaherty 2006).

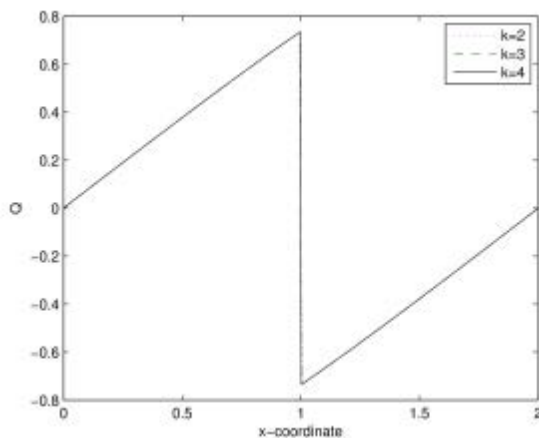


Figure (6.) The numerical solution of example 3. by using TVI-DG with $k = 2$ to 4 at time $t = 1$.

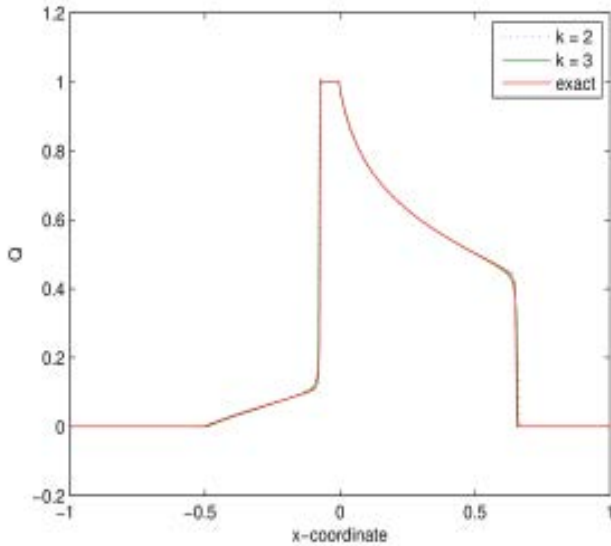


Figure (8). The numerical solution of example 5. by using TVI-DG with $k=2$ and 3 at time $t = 0.4$.

DISCUSSION AND CONCLUSIONS

The transformation of the boundaries integrals into the volume integrals is introduced in this paper under the named TVI-DG method. Thus, there is no need for using the integrals of test functions at the boundaries as in the classical DG method. The totally volume integral discontinuous Galerkin method is used to solve hyperbolic conservation laws. The numerical finding presented that the TVI-DG scheme is very efficient and had lower error magnitude than the other high order schemes as weighted compact scheme.

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التكامل الحجمي لكل الفيض لطريقة جلكين غير المتصلة (TVI-DG) 1- قوانين الحفظ اللامتجهة وأحادية البعد

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المستخلص : يقدم الباحثان في هذا البحث طريقه لتكامل الفيض على كامل الخلية "تكامل بالنسبة للحجم" وذلك في نظرية جلكين غير المتصلة الحدود. إن عملية تحويل التكامل على الحدود للفيض إلى تكامل على كامل الخلية يتم باستخدام نظرية داي فرسن وفي بعض المصادر تسمى نظرية جاوس وبذلك تكون نظرية جلكين غير المتصلة الحدود خالية من التكاملات على حدود الخلية كما هو معروف من خلال نظرية جلكين. في حالة انسياب الموائع وبها موجات صادمة نستخدم طريقة بتروف جلكين المحورة لاستخدامها خلال نظرية جلكين غير المتصلة الحدود وذلك لكبح التذبذبات غير الفيزيائية خلال انسياب الموائع. وقام الباحثان بتطبيق طريقة تكامل الفيض على كامل الخلية على عدة معدلات تفاضلية أحادية البعد مثل معادلة بيرجر. إن نتائج هذا البحث كانت دقيقة جداً عند مقارنتها مع أعمال سابقة وكانت النتائج خالية من التذبذبات في الانسياب به موجات صادمة.

الكلمات المفتاحية: قوانين الحفظ العددية؛ طرق النظام العالي، جلكين غير المتصلة الحدود، نظرية التباعد.

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