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# Green's Function Method for Ordinary Differential Equations

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## Abstract

The analytical solution of non homogeneous boundary value problem in the form:

$$Ly(x) = f(x) \qquad a \le x \le b$$
$$B_1 y(a) = 0$$
$$B_2 y(b) = 0$$

are obtained by using Green's function method. A worked problem is considered for illustration.

## الملخص العربي

في هذا البحث تم حل مسألة القيم الحدية الغير متجانسة "non homogeneous boundary value problem" التي على الصورة:

$$Ly(x) = f(x) \qquad a \le x \le b$$
$$B_1 y(a) = 0$$
$$B_2 y(b) = 0$$

باستخدام دالة جرين "Greeen's function". وقد تم توضيح تطبيق طريقة الحل على مسألة.

**Keywords** Generalized function, Dirac $\delta$-function, Heaviside unit step function, Wronskian, differentiation under the integral sign.

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## 1. Introduction

Differential equations are the group of equations that contain derivatives. An ordinary differential equation (ODE) contains only ordinary derivatives and describes the relationship between these derivatives of the dependent variable, usually taken as *y*, with respect to the independent variable, usually taking as *x*. The solution to such an ordinary differential equation is therefore a function of *x* and is written *y*(*x*). For an ordinary differential equation to have a closed-form solution (Frank 1972; Michael 1998), it must be possible to express *y*(*x*) in terms of the standard elementary functions. The solutions of some differential equations cannot, however, be written in closed form, but only as an infinite power series. Ordinary differential equations may be separated conveniently into different categories according to their general characteristics. The primary grouping adopted here is by the order of the equation. The order of an ordinary differential is simply the order of the highest derivative it contains. Ordinary differential equations may be classified further according to degree. The degree of an ordinary differential equation is the power to which the highest-order derivative is raised, after the equation has been rationalized to contain only integer powers of general function *y*(*x*) that satisfies the equation, it will contain constants of integration (equal to the order) which may be determined by the application of some suitable boundary conditions. When the boundary conditions have been applied, and the constants found, we are left with a particular solution to the ordinary differential equations, which obey the given boundary conditions. In our present paper we introduce Green's function (Donald 1992; Tyn 1973) which will be shown to represent the particular solutions in integral form of boundary value problem for ordinary differential equation.

## 2. Standard Form

The linear ordinary differential equation of the second order has the general form,

$$a_0 y'' + a_1 y' + a_2 y = F(x) \qquad a \le x \le b \qquad (1)$$

where $\boldsymbol{a_0 \neq 0}$, $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$ are continuous functions (coefficients) in the interval $[\boldsymbol{a}, \boldsymbol{b}]$ and $\boldsymbol{F(x)}$ is a piecewise continuous (source function). If $\boldsymbol{F(x) = 0}$, the equation is called homogeneous, otherwise it is non homogeneous.

## 3. Self Adjoint Form

A linear ODE of the second order (1) can be written in the form

$$\frac{d}{dx}\left(p\frac{dy}{dx}\right) + qy = f(x) \quad \text{in} \quad a \le x \le b \qquad (2)$$

i.e.
$$p\frac{d^2y}{dx^2} + \frac{dp}{dx}.\frac{dy}{dx} + qy = f(x)$$

or more compactly as

$$Ly = f(x); \qquad a \le x \le b$$

where $p(x)$ and $q(x)$ are continuous functions and $p(x)$ is continuously differentiable and does not vanish in the interval $[a, b]$, $f(x)$ is a piecewise continuous function, and $L \equiv \frac{d}{dx}\left(p\frac{d}{dx}\right) + q =$ self adjoint operator.

NOTES

3.1. Every linear second order differential equation (1) for which $a_0 \ne 0$ can be expressed (Donald 1992; Michael 1998) in self adjoint form (2).

*Proof:*

multiplying (1) by an integrating factor $\mu(x)$ we get

$$\mu a_0 y'' + \mu a_1 y' + \mu a_2 y = \mu F(x),$$

the equation is in self adjoint form if

$$\mu a_1 = (\mu a_0)' = \mu a_0' + \mu' a_0$$

solving, we get

$$\mu = \frac{1}{a_0} e^{\int \frac{a_1}{a_0} dx}$$

and

$$\frac{d}{dx}\left(e^{\int \frac{a_1}{a_0} dx} y'\right) + \left(\frac{a_2}{a_0} e^{\int \frac{a_1}{a_0} dx}\right) y = \frac{F(x)}{a_0} e^{\int \frac{a_1}{a_0} dx}$$

when we set

$$p(x) = e^{\int \frac{a_1}{a_0} dx}, \quad q(x) = \frac{a_2}{a_0} e^{\int \frac{a_1}{a_0} dx} \quad \text{and} \quad f(x) = \frac{F(x)}{a_0} e^{\int \frac{a_1}{a_0} dx}$$

3.2. When $u(x)$ and $v(x)$ are continuously- differentiable functions on $a \leq x \leq b$ with piecewise continuous second derivatives (Donald 1992; Michael 1998), it is straightforward to show that

$$uLv - vLu = \frac{d}{dx}\{p(uv' - vu')\}.$$

This equation is known as Lagrange's identity, where $L$ is the self adjoint operator.

**4. Linear Boundary Conditions**

Consider conditions of the form

$$\left.\begin{aligned} B_1y &= \propto_1 y(a) + \beta_1\ y'(a) = 0 \\ B_2y &= \propto_2 y(b) + \beta_2\ y'(b) = 0 \end{aligned}\right\} \qquad (3)$$

where the constants $\propto_1$ and $\beta_1$ as also $\propto_2$ and $\beta_2$, are not all zero.
They are called unmixed homogeneous boundary conditions, because one condition is at $x = a$ and the other is at $x = b$ ($a$, $b$ are two boundary points).

**5. Boundary-Value Problems**

In general, the system consists of a second-order differential equation of the form (1) or (2) together with two linear, homogeneous boundary conditions (3), called the boundary-value problem

$$\left.\begin{aligned} a_0y'' + a_1y' + a_2y = F(x) \quad a \leq x \leq b \\ B_1y = \propto_1 y(a) + \beta_1\ y'(a) = 0 \\ B_2y = \propto_2 y(b) + \beta_2\ y'(b) = 0 \end{aligned}\right\} \qquad (4)$$

or

$$\left.\begin{aligned} (py')' + qy = f(x) \qquad a \leq x \leq b \\ B_1y = \propto_1 y(a) + \beta_1\ y'(a) = 0 \\ B_2y = \propto_2 y(b) + \beta_2\ y'(b) = 0 \end{aligned}\right\} \qquad (5)$$

**6. Solutions Of Boundary- Value Problems**

To solve the boundary-value problem (4), we must first find the general solution which is the sum of the complementary function $y_c(x)$ and the particular integral $y_p(x)$.

6.1. Finding The Complementary Function $y_c(x)$

If $a_0, a_1$ and $a_2$ are constants, the standard method for finding $y_c(x)$ is to try a solution of the exponential form, it will contain two arbitrary constants.

6.2. Finding The Particular Integral $y_P(x)$

There is no generally applicable method for finding the particular integral $y_P(x)$ but, for linear ODEs with constant coefficients and a simple forcing function *F(x)*, $y_P(x)$ can often be found (Frank 1972; Michael 1998) by inspection, or by using the differential operators, or applying the method of undetermined coefficients [if $F(x)$ contains only polynomial, exponential, sinusoidal, cosinusoidal]. Two further methods that are useful in finding the particular integral $y_P(x)$ are those based on the variation of parameters, partially known complementary function (reduction of order). The general solution is given by

$$y(x) = y_c(x) + y_p(x) \tag{6}$$

and using the boundary conditions to evaluate the two arbitrary constants.

**7. Solutions Of Boundary-Value Problems Using Green's Function**

The solution of the boundary-value problem (5), is given (Donald 1992; Tyn 1973) in integral form by

$$y(x) = \int_a^b G(x,\xi) f(\xi) d\xi \tag{7}$$

where $\xi$ is the variable of integration, $x$ is a parameter and $G(x,\xi)$ is the Green's function for the boundary value-problem (5), is defined as the solution of

$$LG(x,\xi) = \delta(x-\xi) \tag{7.1}$$

*i.e.* $LG(x,\xi) = 0$ for all $x \neq \xi$

$$B_1 G = \propto_1 G(a,\xi) + \beta_1 G'(a,\xi) = 0 \tag{7.2}$$

$$B_2 G = \propto_2 G(b,\xi) + \beta_2 G'(b,\xi) = 0 \tag{7.3}$$

$G(x,\xi)$ is continuous for all *x* including $\xi$

that is $$G(x,\xi)\Big|_{x=\xi-}^{x=\xi+} = 0$$ (continuity condition) and

$G'(x,\xi)$ is continuous for all *x* except for a discontinuity at $x=\xi$ of magnitude $\frac{1}{p(\xi)}$;

that is $G'(x,\xi)\Big|_{x=\xi-}^{x=\xi+} = \frac{1}{p(\xi)}$ (jump discontinuity condition)

where $\delta(x-\xi)$ is the Dirac delta function (Donald 1992; Glyn 2004; Koshlyakov et al. 1964; Michael 1998; Tyn 1973).

**8. Construction Of Green's Function (Formulas For Green's Functions)**

The associated homogeneous boundary-value problem of (5) is

$$\left.\begin{aligned} Ly = (py')' + qy = 0, \quad a \le x \le b \\ B_1 y = \propto_1 y(a) + \beta_1 y'(a) = 0 \\ B_2 y = \propto_2 y(b) + \beta_2 y'(b) = 0 \end{aligned}\right\} \tag{8}$$

<u>THEOREM</u>

If the associated homogeneous boundary value problem (8) has the trivial solution only (Donald 1992; Tyn 1973), then Green's function exists and is unique.
We summarize these results. The Green's function is given by

$$G(x,\xi) = \frac{1}{pW}\{y_1(x)y_2(\xi)H(\xi-x) + y_1(\xi)y_2(x)H(x-\xi)\}$$

$$= \begin{cases} \frac{1}{pW} y_1(x)y_2(\xi) & for\ \ x<\xi \\ \\ \frac{1}{pW} y_1(\xi)y_2(x) & for\ \ x>\xi \end{cases} \tag{9}$$

where $y_1(x)\, and\, y_2(x)$ are linearly independent solution of $Ly=0$, $W$ is the Wronskian determinant of $y_1(x),\ y_2(x)$ given by

$$W = y_1(x)y_2'(x) - y_2(x)y_1'(x)$$

and $H(x-\xi), H(\xi-x)$ are the Heaviside unit step function (Glyn 2004; Koshlyakov et al. 1964; Michael 1998; Riley et al. 2003; Tyn 1973).

NOTES

8.1. $pW$ is a constant differing from zero

*Proof:*

Since $y_1(x) and\ y_2(x)$ are solutions of the associated homogeneous equation, we have

$$\frac{d}{dx}(py_1') + qy_1 = 0, \quad \frac{d}{dx}(py_2') + qy_2 = 0$$

multiplying the first equation by $y_2$, the second equation by $y_1$ and subtracting, we obtain

$$y_1 \frac{d}{dx}(py_2') - y_2 \frac{d}{dx}(py_1') = 0,$$

which can be written in the form

$$\frac{d}{dx}\{p(y_1y_2' - y_2y_1')\} = 0,$$

integration yields

$$p(y_1y_2' - y_2y_1' = constant\ (independent\ of\ x).$$

8.2. Problems with non homogeneous boundary conditions (Donald 1992; Tyn 1973).

There are two ways to solve this problem, one is to use superposition, and the other is to use Lagrange's identity. Both methods use Green's function for the associated problem with homogeneous boundary conditions.

8.3. Modified (generalized) Green's functions.

When homogeneous problem has nontrivial solution, Green's function does not exist. We define a modified Green's function (Donald 1992; Tyn 1973). Two situations arise, depending on whether the boundary value problem has one or two linearly independent solutions.

**9. Illustrated Example**

Solve the boundary-value problem

$$y'' + 4y = f(x), \qquad 0 < x < 3$$

$$y(0) = 0$$

$$y'(3) = 0$$

when (i) $f(x) = 2x$

(ii) $f(x) = H(x-1) - H(x-2)$

(iii) $f(x) = \delta(x-1)$

SOLUTION

The associated homogeneous system has only the trivial solution, the Green's function for this problem can be obtained from (9).

Solutions of

$$y'' + 4y = 0 \quad i.e. \ \frac{d}{dx}\left(y'\right) + 4y = 0$$

are always of the form

$$y(x) = C_1 sin2x + C_2 cos2x,$$

where $C_1 \ and \ C_2$ are arbitrary constants, solutions that satisfy $y(0) = 0, y'(3) = 0$ respectively, are $y_1(x) = sin2x \ and \ y_2(x) = \cos(6-2x)$, with $p = 1$, $W = y_1 y_2' - y_2 y_1' = -2cos6$, we get

$$G(x,\xi) = \frac{1}{-2cos6}\left[sin2x.\cos(6-2\xi)\,H(\xi-x) + sin2\xi.\cos(6-2x)\,H(x-\xi)\right]$$

with source function $f(x)$, the solution of the boundary value problem is (7)

$$y(x) = \int_0^3 G(x,\xi) f(\xi) d\xi$$

1st case: $f(x) = 2x$, the solution is

$$y(x) = \int_0^3 G(x,\xi).2\xi d\xi = \int_0^x G(x,\xi).2\xi d\xi + \int_x^3 G(x,\xi).2\xi d\xi$$

$$= \int_0^x \frac{1}{-2cos6} sin2\xi.\cos(6-2x).2\xi\ d\xi + \int_x^3 \frac{1}{-2cos6} sin2x.\cos(6-2\xi).2\xi\ d\xi$$

*i.e.* $$y(x) = \frac{x}{2} - \frac{sin2x}{4cos6} \qquad (10)$$

NOTES

9.1. This solution could also be derived simply by finding the general solution of $y'' + 4y = 2x$ and using boundary conditions to evaluate two arbitrary constants, or by using Laplace transform method.

2nd case: $f(x) = H(x-1) - H(x-2)$, the solution is

$$y(x) = \int_0^3 G(x,\xi)[H(\xi-1) - H(\xi-2)]d\xi$$

$$= \int_1^2 G(x,\xi)d\xi, \qquad 1 < \xi < 2$$

when $x \leq 1$

$$y(x) = \int_1^2 \frac{1}{-2cos6} sin2x.\cos(6-2\xi)\, d\xi$$

then

$$y(x) = \frac{sin2x}{4cos6}(sin2 - sin4) \qquad (11)$$

when $1 < x < 2$

$$y(x) = \int_1^x G(x,\xi)d\xi + \int_x^2 G(x,\xi)d\xi$$

$$= \int_1^x \frac{1}{-2cos6} sin2\xi.\cos(6-2x)\, d\xi + \int_x^2 \frac{1}{-2cos6} sin2x.\cos(6-2\xi)\, d\xi$$

*i.e*

$$y(x) = \frac{1}{4} + \frac{1}{4cos6}[sin2x.sin2 - \cos(6-2x)\, cos2] \qquad (12)$$

and when $2 \leq x < 3$

$$y(x) = \int_1^2 \frac{1}{-2cos6} sin2\xi.\cos(6-2x)\, d\xi$$

then

$$y(x) = \frac{\cos(6-2x)}{4cos6}(cos4 - cos2) \qquad (13)$$

9.2. The solutions (11), (12), (13) are not so easily produced using methods from elementary differential equations. They require integration of the differential equation on three separate intervals (0,1), (1,2) and (2,3) then matching of the solution and its first derivative at $x = 1$ and $x = 2$, or by using Laplace transform method.

3rd case: $f(x) = \delta(x-1)$, the solution is

$$y(x) = \int_0^3 G(x,\xi)\delta(\xi-1)d\xi$$

*i.e.* $$y(x) = \frac{1}{-2cos6}\{sin2x.\,cos4.\,H(1-x) + sin2.\cos(6-2x)\,.\,H(x-1)\} \quad (14)$$

$$= \begin{cases} \frac{-1}{2cos6} sin2x.\,cos4 & x < 1 \\ \frac{-1}{2cos6} sin2.\cos(6-2x) & x > 1 \end{cases}$$

9.3. This solution (14) could also be derived very simply by applying Laplace and inverse Laplace transform.

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