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Hybrid Dual Quadrature Rules Combining Open and Closed Quadrature Rules Enhanced by Kronrod Extension or Richardson's Extrapolation for Numerical Integration

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Abstract: Numerical integration is a powerful way to integrate certain categories of integrals, such as those whose closed-form anti-derivative is missing, improper integrals, and tabular data where a function is absent. In this paper, open and closed dual hybrid quadrature rules have been designed for the numerical integration of real definite integrals with either a singular integrand or a non-elementary anti-derivative, respectively. Such quadrature rules couple a Gauss-type rule with a Newton-Cotes-type rule such that both rules are of the same degree of precision, say p to achieve a hybrid rule of a degree of precision greater than or equal to p+2. The open/closed-type hybrid quadrature rule has been constructed as a linear combination between the two-point Gauss-Legendre quadrature enhanced by Kronrod extension and a derivative-based open/closed Newton–Cotes formula yielding a hybrid rule of degree of precision equal to nine. Furthermore, a hybrid quadrature rule was created by merging the numerically enhanced Lobatto-Gauss rule and Bool's rule which was enhanced by Richardson extrapolation. An error analysis analytically confirms that the proposed rules perform better than their ingredients' quadrature rules. The effectiveness of the suggested hybrid rules has been demonstrated with some integral examples that exhibit good agreement with the precise outcomes. An adaptive algorithm has been implemented to enhance the accuracy of the results obtained.

قواعد التربيع المزدوجة والهجينة للتكامل العددي التي تجمع بين قواعد التربيع المفتوحة والمغلقة المُحسَّنة إما بامتداد كرونرود أو الاستكمال الخارجي لريتشاردسون.

المُستخلص: التكامل العددي هو أسلوب فعَّال للتكامل التقريبي لفئةٍ معينةٍ من التكاملات كالتكاملات المُعتلّة أو تلك التي لا يمكن التعبير عن تكاملها في صورة مُغلقة أو بدلالة دوال بسيطة. في هذا البحث تم صياغة بعض قواعد التربيع المزدوجة من النوع المفتوح والمغلق التي تهجِّن بين قاعدة تربيعية من نوع جاوس مع أخرى من نوع نيوتن–كوتس بشرط أن يكون كلاهما بنفس درجة الدقة p . عليه تم تكوين قواعد تربيعة هجينة مفتوحة ومغلقة كتركيبة خطية بين تربيع جاوس– لجندر ذو النقطتين المُحسَّن بامتداد كرونرود وصيغة نيوتن–كوتس المفتوحة/المغلقة القائمة على المشتقات مما ينتج عنه قواعد هجينة ذات دقة من الدرجة التاسعة. أيضاً تم صياغة قاعدة تربيعية هجينة أخرى من النوع المغلق من خلال مزج قاعدة لوباتو–جاوس المُحسَّنة عددياً مع قاعدة بولز المُحسَّنة بالاستكمال الخارجي لريتشاردسون. التحليل الرياضي للخطأ يؤكد أن الطرق الهجينة التي تم توليدها تمتلك دقة أكبر من أو تساوي $p+2$. تم تقديم مجموعة متنوعة من الأمثلة المختلفة للتحقق من كفاءة القواعد الهجينة المُقترحة والتي تظهر توافقًا مُرضِي جدًا مع النتائج المضبوطة. أيضاَ تم تنفيذ خوارزمية تربيعية ملائمة لتحسين دقة النتائج التي تم الحصول عليها.

الكلمات المفتاحية: تربيعة جاوس، قاعدة بولز، امتداد كرونرود، الاستكمال الخارجي لريتشاردسون، صيغ نيوتن كوتس القائمة على المشتقات، تربيعة جاوس المُختلطة، التربيعة العددية.

INTRODUCTION

Numerical integration is a widespread technique to integrate specific categories of integrals with some restrictions, such as integrals that do not possess a closed form,

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elementary anti-derivatives, or improper integrals. One of the very popular tools for numerical integration is the quadrature rules, either Gauss-type or Newton-Cotes-type (Atkinson, 2012; Davis and Rabinowitz, 2012; Burden and Faires, 2005). Quadrature rules are commonly implemented in a variety of applications in physics, engineering, and quantum mechanics. Such quadrature rules are very convenient for computing the anti-derivative of tabular data, which may be encountered in some applications from experiments or measurements where the function is absent. Gauss-type and Newton-Cotes-type quadrature rules are both open, closed, and semi-open types; this should increase their reliability in adopting a variety of integrals with certain constraints. Numerical integration can efficiently integrate improper integrals of different types. Improper integrals over the infinite intervals $(0, \infty)$ or $(-\infty, \infty)$ can be efficiently integrated by Gauss-Lagurre or Gauss-Hermite, respectively (Das and Dash, 2017). Furthermore, such improper integrals can be transformed by a suitable transformation yielding finite limits of integration, then recall a convenient rule for the finite intervals. Gaussian quadrature rules are widely used for integral oscillatory or singular integrands that are encountered in many applications, as evidenced by (Milovanovic, 1998). A comparison between Newton-Cotes quadrature rules and Gaussian quadrature rules is presented in (Sermutlu, 2005) in terms of accuracy, computational effort, and number of integrand evaluations. He claimed that by using low and high-order rules, the quadrature rules of Gauss-Type are superior to the standard Newton-Cotes-Type rules.

The degree of precision of quadrature rules can be improved either by increasing the number of nodes n or decreasing the step size h. However, this may adversely affect the stability of the numerical rule due to the undesirable appearance of negative weights which leads to an ill-conditioned numerical process. Thus, one could resort to the adaptive scheme either globally (on the whole interval of integration) if needed or locally along some sub-regions where the integrand has sharp variation. The mechanism of the adaptive technique is to densely evaluate the integrand in certain sub-intervals where the function experiences large variation to capture the behavior of the integrand in such regions until the termination criterion is met (Dash and Das, 2013a; Dash and Das, 2013b; Dash and Das, 2012). A new set of closed, Mid-point, and open Newton-Cotes quadrature rules were proposed by Burg et. al. (Burg, 2012; Burg, 2013; Zafar, 2014). Such new derivative-based Newton-Cotes formulae require the evaluations of the integrand and its derivative at less abscissa compared to the classical Newton-Cotes rules. They claim that the new scheme of Newton-Cotes formulae yields much better performance compared with the standard Newton-Cotes formulae in terms of accuracy, computational effort, arithmetic operations of the integrand, degree of precision, error terms, and their coefficients.

The degree of precision of n-point the Gaussian rule is $(2n-1)$, that is this rule should exactly integrate any polynomials of a degree less than or equal to $(2n-1)$. The n-point Newton-Cotes quadrature rules are of the degree of precision $(n-1)$ if n is even, and of the degree of precision n if n is odd. Recently, a numerical enhancement was proposed by (Babolian et al., 2005; Masjed et. al., 2005) to increase the accuracy by two for the Gauss-Legendre and Gauss-Radau quadrature rules. Furthermore, a numerical enhancement of the Gauss-Lobatto quadrature rule was proposed by (Eslahchi et. al., 2005), and they claim that they obtained better approximate results than those obtained by the corresponding standard Gauss-Lobatto quadrature rule. Moreover, such a technique has been adopted for the open, closed, and semi-open Newton-Cotes formulae, respectively (Dehghan et. al., 2006; Dehghan et. al., 2005a; Dehghan et. al., 2005b).

It is worth emphasizing that those Gaussian quadrature rules are stable because all the weights are positive. Gauss-Legendre, Hermite, and Lagurre quadrature rules are of open type, whereas Gauss-Lobatto and right/left Gauss-Radaua are respectively of closed and semi-open/closed type. A drawback of Gauss-type rules is that they are not progressive, that is, Gaussian rules of differentiation, and have no nodes in common except at the midpoint. To overcome the non-progressive issue related to Gaussian quadrature rules, Kronrod (Kronrod, 1965 a; Kronrod, 1965 b; Walter, 1988) established his extension to the Gauss-Legendre and Lobatto quadrature rules, both of which have a weight function ($w(x) = 1$). The Kronrod extension optimally adds $(n + 1)$ abscissas to the n-point Gaussian, yielding a more accurate $(2n + 1)$-point Kronrod–Legendre Gauss pair quadrature rule. In contrast, for other Gaussian quadrature rules, such as Hermite-Gauss and Lagurre-Gauss, there is no real Kronrod extension (Kahaner, 1978), but sub-optimal Kronrod extensions can be gained with a degree of precision less than $(3n + 1)$ (Begumisa and Robinson, 1991). Additionally, Patterson (Patterson, 1968 a; Patterson, 1968 b) enlarged the idea of Kronrod extension by adding $(2n + 2)$ points to the $(2n + 1)$-point Kronrod–Gauss pair to achieve a progressive rule of $(6n + 4)$ as a degree of precision.

The accuracy of the numerical quadrature rules can be enhanced by adopting some reliable approaches such as Richardson's extrapolation (Burden and Faires 2005) and the Kronrod extension which respectively rely on the trapezoidal rule and quadrature rule as a fundamental rule. Richardson extrapolation (Zlatev et. al., 2018) is a powerful technique to enhance the accuracy of approximation numerical tools that deal with a parameter say the step size h such as numerical integration, numerical differentiation, numerical methods for solving ordinary and partial differential equations such as Runge-Kutta and finite difference methods respectively. The advantage of implementing the Richardson extrapolation to quadrature rule is to gain a higher accuracy relying on low-order rules and can be efficiently incorporated into Gauss-Type (Mohanty, 2020; Jena and Dash, 2011) and Newton-Cotes-Type quadrature rules (Jena and Dash, 2011).

Furthermore, a simple approach was first proposed by (Das and Pradhan, 1996) combining a pair of quadrature rules of the same degree of precision, say p, producing a mixed quadrature rule of better accuracy, usually $(p + 2)$. They combine the 3-point Gauss-Legendre with Simpson's 1/3 quadrature rule. It is worth emphasizing that, the formula of the mixed quadrature rule does not involve any extra sampling of the integrand, it only linearly couples the ingredient rules to gain better accuracy. Other formulations of mixed rules are found blending different Gauss-type with Newton-Cotes-Type quadrature rules for approximate evaluation of real definite integral and also for analytic functions (Tripathy et. al., 2015; Patra et. al., 2018; Mohanty, 2020). Such mixed quadrature rules have been implemented to solve singular integral equations in electromagnetic field problems (Jena and Nayak, 2015).

In this paper, three dual hybrid quadrature rules have been constructed for the numerical integration of real definite integrals with singular integrands or non-elementary anti-derivatives. Such quadrature hybridizes between a Newton-Cotes-type formula and a Gauss-type formula enhanced either by Kronrod extension or Richardson extrapolation, both of which have the same degree of precision. This paper is structured as follows: The relevant literature review is presented in Section 1. In Section 2, we introduce some basic definitions. In Section 3, the Kronrod extension of the two-point Gauss-Legendre quadrature rule has been constructed. The open and closed type hybrid rule coupling the Gauss-Kronrod quadrature rule with a derivative-based open and closed type Newton-Cotes rule, respectively, has been formulated in sections 3 and 4. The further closed-type hybrid rule was formulated in Section 5 by combining a numerically enhanced Gauss-Lobatto quadrature rule with Bool's Rule enhanced by Richardson extrapolation. For the sake of verification of the proposed hybrid rules of both types, some numerical results are shown in Section 6, followed by a discussion in Section 7. Finally, a conclusion is drawn in Section 8.

Basic Definitions

Here we introduce some basic definitions that we need throughout the paper.

Definition 1 [1]: An n-point Gaussian-quadrature rule is defined by the formula,

$$I_n(f) = \int_a^b f(x)\,dx \cong \sum_{i=1}^{n} w_i f(x_i) + EI_n(f), \quad (1)$$

where the points x_i are the quadrature points and are known as nodes or abscissas, the factors w_i are the corresponding weights, and $EI_n(f)$ is the error of the rule (1). The quadrature rule (1) is based on polynomial interpolation. The mechanism of the Gauss quadrature is based on the precision concept, that is, the quadrature rule is exact for polynomials of degrees less than or equal to $2n-1$. That is the formula (1) exactly integrates n monomial functions $x^i, i = 0,1,2,\dots,n$. Thus we obtain a non-linear system of moment equations that can be solved, yielding the nodes and the corresponding weights.

Definition 2 (Degree of Precision): The degree of precision of the n-point Gaussian-quadrature rule (1) is defined by the degree of the polynomial such that the error $EI_n(P_n) = 0$. Thus the quadrature rule (1) is exact for all polynomials of degree $\le n$, and the error $EI_n(P_n) \neq 0$ for $i = n+1, n+2, \dots$

It is worth emphasizing that for the Newton-Cotes quadrature rules, the equal-distance nodes are known and the weights are unknowns and need to be determined by solving a Vandermonde system, whereas for the Gaussian quadrature rules, the nodes and the weights are both unknowns.

The two-point Gauss-Legendre quadrature rule is given as,

$$I_{GL2}(f) = h\left[f\left(\mu - \frac{h}{\sqrt{3}}\right) + f\left(\mu + \frac{h}{\sqrt{3}}\right)\right], \quad (2)$$

where $h = \frac{b-a}{2}$ and throughout the paper $\mu = \frac{a+b}{2}$ denotes the mid-point of the reintegration interval.

$$I_{Exact}(f) = \int_a^b f(x)\,dx = I_{GL2}(f) + E_{GL2}(f), \quad (3)$$

Where $E_{GL2}(f)$ is the truncation error of the two-point Gauss-Legendre quadrature. The error can be derived by polynomials interpolation or by Taylor expansion of the functions involved in $I_{GL2}(f)$ about the mid-point μ of the integration interval to yield,

$$E_{GL2}(f) = I_{Exact}(f) - I_{GL2}(f) = \frac{h^5}{135} f^{(4)}(\mu) + \frac{1016h^7}{675 \times 7!} f^{(6)}(\mu) + \cdots$$

The degree of precision of the two-point Gauss-Legendre quadrature rule $I_{GL2}(f)$ is three, that is, it should exactly integrate polynomials of degree up to three.

Definition 3 (Stability of Quadrature Rule): If all weights in the formula (1) are non-negative, then the rule is stable and

$$\lambda = \sum_{i=1}^{n} |w_i| = b - a,$$

where λ is known as the absolute condition number of the quadrature rule.

Definition 4 (Progressive Quadrature Rule): A quadrature rule is called progressive if the nodes for I_{n_1} are also nodes for the successive rule I_{n_2} where $n_2 > n_1$.

The quadrature rule has this outstanding feature that significantly reduces the computational effort for successive quadrature rules by keeping the arithmetic operations that are involved in integrating the integrands to a minimum. Unfortunately, Gauss-Type rules are not progressive; that is Gaussian rules of different n have no nodes in common except at the midpoint. To overcome this issue, Kronrod 1965 established his progressive extension to the Gauss-Legendre quadrature as shown next.

Kronrod Extension of Two-Point Gauss-Legendre Quadrature Rule

The Kronrod extension (Walter, 1988) optimally adds $(n+1)$ abscissas to the n-point Gaussian rule yielding an $(2n+1)$-point Kronrod–Legendre Gauss pair quadrature rule of $(3n+1)$ or $(3n+2)$ as degree of precision depending on whether n is even or odd respectively. The $(2n+1)$-point Kronrod–Legendre-Gauss pair quadrature $I_{2n+1}(f)$ is progressive as it only requires sampling of the integrand at the new $(n+1)$ points.

Now we show how to enhance the degree of precision of $I_{GL2}(f)$ seeking its Kronrod extension. Such extension can be achieved by adding three new abscissa to the $I_{GL2}(f)$ in equation (2); that is, we have,

$$I_{KGL2}(f) = \sum_{i=1}^{5} c_i f(x_i),$$

where, $x_2 = \mu - \frac{h}{\sqrt{3}}$, and $x_4 = \mu + \frac{h}{\sqrt{3}}$.

To force this quadrature rule to exactly integrate polynomials of degree $3n+1 = 7,$ where $n = 2$, we need to consider the monomial functions $f(x) = x^j, j = 0,1,2, \ldots .3n+1$. Thus we have an algebraic non-linear system with eight unknowns $c_1, c_2, c_3, c_4, c_5, x_1, x_3, x_5$, and eight-moment equations, which can be solved to obtain the Kronrod extension of the Gauss-Legendre quadrature rule $I_{KGL2}(f)$ as,

$$I_{KGL2}(f) = \frac{h}{495}\left\{\begin{matrix} 98\left[f\left(\mu - \sqrt{\frac{6}{7}}h\right) + f\left(\mu + \sqrt{\frac{6}{7}}h\right)\right] + 308f(\mu) \\ +243\left[f\left(\mu - \frac{1}{\sqrt{3}}h\right) + f\left(\mu + \frac{1}{\sqrt{3}}h\right)\right] \end{matrix}\right\}. \quad (4)$$

This formula can be written as,

$$I_{Exact}(f) = I_{KGL2}(f) + E_{KGL2}(f). \quad (5)$$

where $E_{KGL2}(f)$ is the truncation error of the Kronrod-Gauss quadrature rule and can be computed by Taylor expansions of the functions involved in $I_{KGL2}(f)$ to yield,

$$E_{KGL2}(f) = -\frac{8h^9 f^{(8)}(\mu)}{245 \times 9!} - \frac{5672 h^{11} f^{(10)}(\mu)}{7^3 \times 11!} - \cdots$$

Thus, the Kronrod extension of the two-point Gauss-Legendre rule considerably enhanced the degree of precision from three to seven, and the local truncation error is of the ninth order. It is worth mentioning that the Gauss-Legendre quadrature rule is of open type because all of its nodes are interior points and usually cluster near the endpoints of the integration interval. Efficient computation of improper integrals with singular integrands can be achieved by this rule, either alone or in conjunction with other open-type Newton-Cotes quadrature formulas, as demonstrated later. As we have just shown, the Gauss-type quadrature rules can be enhanced by their Kronrod extension, whereas the adaptive quadrature rule can be utilized to enhance the approximate results obtained by the Newton-Cotes-type quadrature rules. Next, we briefly give an overview of the adaptive quadrature rule.

Adaptive Algorithm

A mathematical integration technique called adaptive quadrature (Stoer and Bulirsch, 1992; Burden and Faires, 2005) is utilized to approximate the definite integral. Dynamic adjustments are made to the subintervals and the number of evaluation points by adaptive quadrature based on the local behavior of the integrand. This aids in obtaining a more precise approximation, especially in situations where functions change rapidly. The adaptive quadrature algorithm usually consists of the following steps:

Initial setup: Break up the integration interval into multiple subintervals and approximate the integral for each subinterval.

Error Estimation:

- Compare the results obtained with the quadrature rule to estimate the error in each subinterval.
- Determine which subintervals have a significant impact on the error.

Refinement:

- Reduce the error by dividing the most significant subintervals into smaller subintervals.

- Perform the integration process again for the subintervals that have been refined.

Termination: Keep repeating the process of error estimation, adaptability, and refinement till a specific level of accuracy is reached or until a termination criterion is satisfied.

Formulation of the Open-Type Hybrid rule coupling the Gauss-Kronrod rule with a Derivative-Based Open-Type Newton-Cotes rule.

Here, we show how to couple two quadrature rules that both have the same degree of precision, say seven, to yield a hybrid rule of the same degree of precision, nine. The ingredients of the open-type hybrid rule are the Kronrod extension of the two-point Gauss-Legendre rule and a derivative-based open-type Newton-Cotes rule for n=3 which is given by Zafar (Zafar 2014),

$$I_{Exact}(f) = \frac{h}{224}\left\{\begin{matrix} 1805[f(x_1)+f(x_2)] - 1245[f(x_0)+f(x_3)] \\ +h\begin{bmatrix} \frac{6605}{9}[f'(x_3)-f'(x_0)] + \\ 1315[f'(x_2)-f'(x_1)] \end{bmatrix} \end{matrix}\right\} + \frac{5951\,h^9}{1016064} f^{(8)}(\mu) + \cdots,$$

where the quadrature points are,

$$x_i = a + (i+1)h, \qquad i = 0,1,2,\dots n,$$

and the step size is defined as:

$$h = \left(\frac{b-a}{n+2}\right).$$

This formula can be rewritten as,

$$I_{Exact}(f) = I_{NDO}(f) + E_{NDO}(f), \quad (6)$$

where,

$$I_{NDO}(f) = \frac{h}{224}\left\{\begin{matrix} 1805[f(x_1)+f(x_2)] - 1245[f(x_0)+f(x_3)] \\ +h\left[\frac{6605}{9}[f'(x_3)-f'(x_0)] + 1315[f'(x_2)-f'(x_1)]\right] \end{matrix}\right\}, \quad (7)$$

and,

$$E_{NDO}(f) = \frac{5951\,h^9}{1016064} f^{(8)}(\mu) + \mathcal{O}(h^{11}).$$

It should be noted that the new derivative-based Newton-Cotes formula I_{NDO} has a degree of precision of seven, the local truncation error is of ninth order, and it only needs four interior quadrature points. Thus, I_{NDO} requires the evaluations of the integrand and its first derivative at a lower number of abscissas compared to the classical Newton-Cotes rules.

Now, multiplying equations (5) and (6), respectively, by $\frac{29755}{2}$ and then adding the resulting equations yields the open-type hybrid quadrature rule as follows:

$$I_{OHR}(f) = \frac{70}{1041441}\left[\frac{29755}{2} I_{KGL2}(f) + \frac{8}{35} I_{NDO}(f)\right], \quad (8)$$

where $I_{KGL2}(f)$ and $I_{NDO}(f)$ are respectively, given by equations (6) and (7). The local truncation error of $E_{OHR}(f)$ is of ninth order, that is,

$$E_{OHR}(f) = \mathcal{O}(h^{11}). \quad (9)$$

We already know that the Gauss-Legendre quadrature rule is of open type because all of its nodes are interior points of the integration interval. Combining this rule with a closed-type Newton-Cotes rule can enable the computation of integrals without a closed-form anti-derivative or a non-elementary anti-derivative, as demonstrated below.

Formulation of a Closed-Type Hybrid Rule Coupling the Gauss-Kronrod Rule with a Derivative-Based Closed-Type Newton-Cotes Rule.

Furthermore, we can formulate a closed-type quadrature rule with a degree of precision of seven. Let us recall a derivative-based closed-type Newton-Cotes rule for $n = 3$ (Burg 2012) given as

$$I_{Exact}(f) = \int_a^b f(x) = \frac{h}{224}\left\{\begin{array}{c} 93[f(a)+f(b)] + 243[f(x_1)+f(x_2)] \\ +\frac{h}{5}\left[57[f'(a)-f'(b)] + 81[f'(x_2)-f'(x_1)]\right] \end{array}\right\} + \frac{9\, h^9 f^{(8)}(\mu)}{313 \times 600} + \cdots, \quad (10)$$

where the quadrature points $x_i = a + ih,\ i = 0,1,2,\ldots n,$ and the step size is $h = \frac{b-a}{n}$, with degree of precision seven. The formula (10) can be rewritten as,

$$I_{Exact}(f) = I_{NDC}(f) + E_{NDC}(f), \quad (11)$$

where the error is given as:

$$E_{NDC}(f) = \frac{9\, h^9 f^{(8)}(\mu)}{313 \times 600} + +\mathcal{O}(h^{11}).$$

In a similar fashion to the formulation in Section 3, one can easily obtain the following linear combination of equation (6) with equation (11) yielding a closed-type hybrid quadrature rule as,

$$I_{CHR}(f) = \frac{15337}{167011}\left[\frac{3402}{313} I_{KGL2}(f) + \frac{1}{49} I_{NDC}(f)\right],$$

where $I_{KGL2}(f)$ and $I_{NDC}(f)$ are respectively given by equations (6) and (10). The corresponding truncation error of $I_{CHR}(f)$ is,

$$E_{CHR}(f) = \mathcal{O}(h^{11}). \quad (12)$$

Next, we formulate a closed-type hybrid rule by blending two closed quadrature rules.

Formulation of The Closed -Type Hybrid Rule Coupling the Numerically Enhanced Gauss-Lobatto Quadrature Rule with Bool's Rule Enhanced by Richardson Extrapolation.

First, let us start with the Gauss-Lobatto quadrature rule.

1. Numerically Enhanced Gauss-Lobatto Quadrature Rule

The standard n-point Gauss-Lobatto quadrature rule is given by the following formula,

$$\int_a^b f(x)\,dx \approx I_{Lob}(f) = c_1 f(a) + \sum_{i=2}^{n-1} c_i\, f(x_i) + c_n f(b), \quad (13)$$

where the abscissas x_i are the $(i-1)$th zero of the $P'_{n-1}(x)$, and $P_n(x)$ is the n^{th} degree Legendre polynomial. This rule is closed because both of the endpoints a, b are also taken as quadrature points, and the degree of precision of this rule is $(2n-3)$.

A numerical enhancement of the Gauss-Lobatto quadrature rule (13) was proposed by (Eslahchi et. al. 2005), they claim that their approach yields better approximate results than those obtained by the corresponding standard Gauss-Lobatto quadrature rule. The core idea of their approach is to consider the end-points of the integral as parameters and that the monomial basis functions x^i is extended from $i = 0,1,2,\ldots,2n+1$ (for the standard Gauss-Lobatto) to $i = 0,1,2,\ldots,2n+3$. Thus the proposed approach is approximately exact for polynomials of degree up to $2n+3$. That is, they proposed the following system,

$$\int_a^b x^i dx = \frac{b^{i+1} - a^{i+1}}{i+1} = \sum_{k=1}^{n} w_k\, x_k^i + \alpha a^i + \beta b^i,$$

for $i = 0,1,2,\ldots,2n+3,$ and the notations $a, b, \alpha, \beta, x_1, x_2, \ldots, x_n, w_1, w_2, \ldots, w_n$ are all unknowns, resulting in a non-linear system that has no analytic solution, but a numerical solution can be found. Thus, all the abscissas, the optimal location of endpoints, and the corresponding weights only have numerical values that are tabulated in (Eslahchi et. al

2005). Thus, one needs to rescale the original integral to fit the new optimal endpoints by the following transformation (Eslahchi et al., 2005),

$$\int_{\delta}^{\tau} f(x)dx = \int_{a}^{b} \psi(x)dx,$$

where,

$$\psi(x) = \left(\frac{\tau-\delta}{b-a}\right) f\left(\frac{(\tau-\delta)x + b\delta - a\tau}{b-a}\right).$$

A numerically enhanced Gauss-Lobatto quadrature rule for $n = 1$ takes the numerical form within a tolerance 10^{-7}(Eslahchi et. al 2005),

$$\int_{a}^{b} f(x)\,dx \approx I_{NLob}(f) = 4.7 \times 10^{-9} + 10^{-3}\begin{bmatrix} 9.3801\,\psi(a) + 9.3802\,\psi(b) \\ +37.5206\,\psi(2.5022\times 10^{-3}) \end{bmatrix}, \quad (14)$$

where the optimal locations of the endpoints are:

$a = -2.789016 \times 10^{-2}, b = 2.839074 \times 10^{-2}$

Here, we show how to couple two closed quadrature rules, both having the same degree of precision, to yield a more accurate hybrid rule. The ingredients of the hybrid rule are a numerically improved Gauss-Lobatto quadrature rule, and Bool's rule enhanced by Richardson extrapolation.

2. Bool's Rule Enhanced by Richardson Extrapolation.

The closed Newton-Cotes quadrature rule for $n = 4$ is (five abscissas) known as Bool's rule, and is defined by the following formula,

$$\int_{a}^{b} f(x) \approx I_{Bool}(f) = \frac{2h}{45}\left\{\begin{matrix} 7[f(a)+f(b)] + 32[f(x_1)+f(x_3)] \\ +12f(x_2) \end{matrix}\right\} - \frac{8\,h^7}{945} f^{(6)}(\mu). \quad (15)$$

The corresponding error is,

$$E_{Bool}(f) = -\frac{(2h)^7}{21 \times 6!} f^{(6)}(\mu) + \cdots,$$

where the quadrature points $x_i = a + ih,\ i = 0,1,2,\dots n,$ and the step size is $h = \frac{b-a}{n}$, with degree of precision five. Here we show how to enhance the Bool's rule by Richardson extrapolation. The mechanism of Richardson extrapolation is to begin with an initial approximation at a certain level of refinement, and then compute a successive approximation using a finer level of refinement. Finally, apply the following Richardson extrapolation formula (Zlatev et. al. 2018) yielding an enhanced accuracy of the approximated integral,

$$\int_{a}^{b} f(x)\,dx \approx I_4^{(k)} = \frac{4^k I_{2n}^{(k-1)} - I_n^{(k-1)}}{4^k - 1}, \quad (16)$$

for $n \geq 2^k, k \geq 1$.

Starting with $k = 1$ in (16) we have

$$I_4^{(1)} = \frac{4I_8^{(0)} - I_4^{(0)}}{3}, \quad (17)$$

where,

$$I_4^{(0)} = I_{Bool}(f).$$

Now for $I_8^{(0)}$ we have nine points, thus

$$I_8^{(0)} = \frac{h}{45}\left\{\begin{matrix} 7[f(a)+f(b)] + 32\begin{bmatrix} f\left(a+\frac{h}{2}\right) + f\left(a+\frac{3h}{2}\right) \\ +f\left(a+\frac{5h}{2}\right) + f\left(a+\frac{7h}{2}\right) \end{bmatrix} \\ +12[f(a+h)+f(a+3h)] + 14f(a+2h) \end{matrix}\right\},$$

By substituting $I_4^{(0)}$ and $I_8^{(0)}$ into equation (17), we obtain

$$I_{RBool}(f) = I_4^{(1)}(f) = \frac{2h}{135}\left\{\begin{matrix} 7[f(a)+f(b)] - 8[f(a+h)+f(a+3h)] \\ +64\begin{bmatrix} f\left(a+\frac{h}{2}\right) + f\left(a+\frac{3h}{2}\right) \\ +f\left(a+\frac{5h}{2}\right) + f\left(a+\frac{7h}{2}\right) \end{bmatrix} \end{matrix}\right\},$$

where the truncation error of the enhanced quadrature rule $I_4^{(1)}$ can be computed by Taylor expansions of the functions involved in $I_4^{(1)}$ to yield,

$$EI_4^{(1)}(f) = \frac{5\,(2h)^7}{336 \times 6!} f^{(6)}(\mu) + \frac{277\,(2h)^9}{11520 \times 8!} f^{(8)}(\mu) + \cdots,$$

Comparing the error $EI_4^{(1)}(f)$ with $E_{Bool}(f)$, one could notice that the magnitude of the coefficient of the leading term of $EI_4^{(1)}$ has decreased by an amount of $\frac{5}{16}$ compared to the corresponding term in $E_{Bool}(f)$.

For $k = 2$ in (16), we have

$$I_4^{(2)} = \frac{4I_8^{(1)} - I_4^{(1)}}{15}, \qquad (18)$$

where,

$$I_8^{(1)} = \frac{h}{135}\left\{7[f(a)+f(b)] + 32\begin{bmatrix} f\left(a+\frac{h}{4}\right)+f\left(a+\frac{3h}{4}\right)+f\left(a+\frac{5h}{4}\right)+ \\ f\left(a+\frac{7h}{4}\right)+f\left(a+\frac{9h}{4}\right)+f\left(a+\frac{11h}{4}\right) \\ +f\left(a+\frac{13h}{4}\right)+f\left(a+\frac{15h}{4}\right)\end{bmatrix} + 12\left[f\left(a+\frac{h}{2}\right)+f\left(a+\frac{3h}{2}\right)+f\left(a+\frac{5h}{2}\right)+f\left(a+\frac{7h}{2}\right)\right] + 14[f(a+h)+f(a+2h)+f(a+3h)]\right\}.$$

Substituting $I_4^{(1)}$ and $I_8^{(1)}$ into equation (18) yields,

$$I_4^{(2)} = \frac{2h}{2025}\left\{77[f(a)+f(b)] + 384\begin{bmatrix} f\left(a+\frac{h}{4}\right)+f\left(a+\frac{3h}{4}\right)+f\left(a+\frac{5h}{4}\right) \\ +f\left(a+\frac{7h}{4}\right)+f\left(a+\frac{9h}{4}\right) \\ +f\left(a+\frac{11h}{4}\right)+f\left(a+\frac{13h}{4}\right)+f\left(a+\frac{15h}{4}\right)\end{bmatrix} + 80\left[f\left(a+\frac{h}{2}\right)+f\left(a+\frac{3h}{2}\right)+f\left(a+\frac{5h}{2}\right)+f\left(a+\frac{7h}{2}\right)\right] + 176[f(a+h)+f(a+2h)+f(a+3h)]\right\},$$

the truncation error of the enhanced quadrature rule $I_4^{(2)}$ and can be computed similarly to the error of $I_4^{(1)}$, so

$$EI_4^{(2)}(f) = -\frac{9\,(2h)^7}{8960 \times 6!} f^{(6)}(\mu) - \frac{421\,(2h)^9}{245760 \times 8!} f^{(8)}(\mu) - \cdots,$$

A similar process can be followed for $k = 3$. To prevent repetition, in a similar analogy to the derivation in Sections 3 and 4, one could linearly combine the numerically enhanced Gauss-Lobatto quadrature rule with Bool's rule enhanced by Richardson extrapolation, yielding a closed-type quadrature rule $I_{CBoLob}(f)$, and the corresponding truncation error is,

$$E_{CHR}(f) = \mathcal{O}(h^9). \qquad (19)$$

RESULTS

Some integral examples are presented in Table (1) to verify the efficiency of the open-type mixed quadrature rule $I_{OHR}(f)$. For example, the following logarithmic integral has non-elementary anti-derivative as:

$$I_1 = \int_1^2 \ln[\ln(x)]dx = -\mathrm{Li}(2) + \gamma + 2\log(\log 2) \approx -1.20097,$$

where $\mathrm{Li}(x) = \int_0^x \frac{dy}{\ln(y)}$ is the logarithmic integral function, and γ is the Euler–Mascheroni constant; thus such an integral only has a numerical value. Other integral examples of singular-kernel are presented, such as elliptic integral I_2, exponential integral I_3, error function I_6, Dirichlet integral I_7 (Kober, 1940), and incomplete gamma function I_4. Concerning the Dirichlet integral I_7, we need the variable transformation $w = e^{-x}$ to transform the indefinite integral I_7 to a definite integral as follows:

$$I_7 = \int_0^\infty \frac{\sin x}{x} dx \overset{x=\ln\left(\frac{1}{w}\right)}{\Longleftrightarrow} \int_0^1 \frac{\sin\left[\ln\left(\frac{1}{w}\right)\right]}{w\ln\left(\frac{1}{w}\right)} dw.$$

Now, we recall the following transformation:

$$\int_a^b f(w)dw = \int_{-1}^{1} f\left[\frac{(b-a)x+b+a}{2}\right]\left(\frac{b-a}{2}\right)dx,$$

Thus, one has

$$\int_0^\infty \frac{\sin x}{x}dx = \int_{-1}^{1} \frac{\sin\left[\ln\left(\frac{2}{x+1}\right)\right]}{(x+1)\ln\left(\frac{2}{x+1}\right)}dx.$$

Also, some integral examples of the closed-type mixed quadrature rule are presented in Table (2) to verify the efficiency of the closed-type mixed quadrature rule $I_{CHR}(f)$. For example, the following logarithmic integral has non-elementary anti-derivative:

$$I_1 = \int_1^2 e^{e^x}\,dx = E\mathrm{i}(e^2) - E\mathrm{i}(e) \approx 255.676,$$

where $\mathrm{Ei}(x) = -\int_{-x}^{\infty} \frac{e^{-y}}{y}dy$ is the exponential integral, thus the integral I_1 only has an approximate value. These approximate integral values are used for verification purposes by comparing them with the numerical results obtained by the proposed rules and are referred to as near-exact values. Other integral examples of non-elementary anti-derivatives are also presented, such as the Gaussian integral I_2(encountered in probability density), the sine integral, and the exponential integral I_4.

DISCUSSION

For numerical computations, we build up some numerical routines by Mathematica 3.1. software. Table (1) shows the approximate values of some improper integrals, either with singular integrands or with infinite intervals of integration. With a small number of abscissas, the observed accuracy is quite good and very satisfactory. It should be noted that the relative errors related to the approximate results shown in Tables (2) and (3) are much smaller than those in Table (1). This variance between both categories can be traced back to the fact that the integrands in Table (2) are quite well-behaved functions, unlike the integrands in Table (1) which can be considered bad-behaved functions. For instance, the Dirichlet integral I_7 in Table (1) has a singularity at both endpoints of the integration interval. Also, the integrand of I_1 in Table (2) experiences very sharp variations, especially in the sub-region (1.5,2) as shown in Figure (1). Thus, to achieve higher accuracy for $I_{OHR}(f)$, the rule $I_{NDO}(f)$ needs to be enhanced by an adaptive algorithm. Thus, one urgently needs a local-adaptive algorithm for $I_{NDO}(f)$ and $I_{NDC}(f)$ to conveniently capture the integrand behavior rather than only relying on four nodes. However, such integrand behavior will be inherited in the adaptive algorithm; the adaptive quadrature algorithm for integrals in Table (1) may suffer from slow convergence and thus need quite a few iterations, as shown in Table (4). Tables (4) and (5) show that the approximate results agree with the near-exact ones up to four digits as we set up the termination criterion of the adaptive algorithm to 10^{-5}.

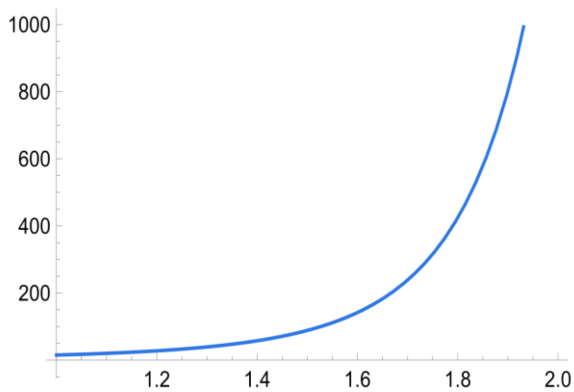

Figure (1): The function $f(x) = e^{e^x}$ along the interval [0,2]

Table: (1). Numerical results computed by the open-type hybrid quadrature rule $I_{OHR}(f)$ compared with its constituent rules $I_{KGL2}(f)$ rules and $I_{NDO}(f)$.

Integral/Transformed Int.	Near-Exact	$\boldsymbol{I_{KGL2}(f)}$	$\boldsymbol{I_{NDO}(f)}$	$\boldsymbol{I_{OHR}(f)}$	Relative Error
$\boldsymbol{I_1 = \int_1^2 \ln[\ln(x)]dx}$ $\boldsymbol{I_1 = \frac{1}{2}\int_{-1}^{1} \ln\left[\ln\left(\frac{x+3}{2}\right)\right] dx}$	≈ -1.20097	-1.186269827214	-1.150942419465	-1.186203619128	$E_{KGL2} = 1.224350377214 \times 10^{-2}$ $E_{NDO} = 4.165913561106 \times 10^{-2}$ $E_{OHR} = 1.2298632432924 \times 10^{-2}$
$\boldsymbol{I_2 = \int_0^1 \sqrt{1-x^4}dx}$ $\boldsymbol{I_2 = \frac{1}{8}\int_{-1}^{1} \sqrt{16-(x+1)^4}dx}$	≈ 0.8740191847	0.8747043456216	0.8787134586882	0.8747118592129	$E_{KGL2} = 7.839197005632 \times 10^{-4}$ $E_{NDO} = 5.37090490231107 \times 10^{-3}$ $E_{OHR} = 7.925162982014 \times 10^{-4}$
$\boldsymbol{I_3 = \int_1^{\infty} \frac{e^{-x}}{x} dx}$ $\boldsymbol{I_3 = \int_{-1}^{1} \frac{e^{-\left(\frac{2}{1+x}\right)}}{1+x} dx}$	≈ 0.219384	0.21841054884110	0.23028552760062	0.21843280407190	$E_{KGL2} = 4.43690444836492 \times 10^{-3}$ $E_{NDO} = 4.9691848380501 \times 10^{-2}$ $E_{OHR} = 4.335460234298 \times 10^{-3}$
$\mathbf{I_4} = \boldsymbol{\int_0^{\infty} e^{-x}\, x\, dx}$ $\mathbf{I_4} = \boldsymbol{\frac{1}{2}\int_{-1}^{1} \ln\left(\frac{2}{x+1}\right) dx}$	$\Gamma(2) = 1$	0.9852958834667177	0.9499682436581703	0.9852296749459611	$E_{KGL2} = 1.47041165332823 \times 10^{-2}$ $E_{NDO} = 5.00317563418297 \times 10^{-2}$ $E_{OHR} = 1.477032505403886 \times 10^{-2}$
$\boldsymbol{I_5 = \int_0^{\infty} \frac{e^{-x}}{1+x^2} dx}$ $\boldsymbol{I_5 = 2\int_{-1}^{1} \frac{\left[e^{-\left(\frac{1+x}{2}\right)} + e^{-\left(\frac{2}{1+x}\right)}\right]}{x^2+2x+5} dx}$	≈ 0.62145	0.621317534623	0.6232082213946	0.6213210780121	$E_{KGL2} = 2.125507968272 \times 10^{-4}$ $E_{NDO} = 2.8298305931829 \times 10^{-3}$ $E_{OHR} = 2.0684898452106 \times 10^{-4}$
$\boldsymbol{I_6 = Erfc(1) = \frac{1}{\sqrt{2\pi}}\int_1^{\infty} e^{-x^2/2}\, dx}$ $\boldsymbol{I_6 = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} \frac{e^{-\left[\ln\left(\frac{2e}{x+1}\right)\right]^2/2}}{1+x} dx}$	≈ 0.15865525	0.1587331354611	0.15817808821860	0.1587320952315	$I_{KGL2} = 4.908852857186 \times 10^{-4}$ $I_{NDO} = 3.0075632607556 \times 10^{-3}$ $I_{OHR} = 4.8432874516111 \times 10^{-4}$
$\boldsymbol{I_7 = \int_0^{\infty} \frac{\sin x}{x} dx}$ $\boldsymbol{I_7 = \int_{-1}^{1} \frac{\sin\left[\ln\left(\frac{2}{x+1}\right)\right]}{(x+1)\ln\left(\frac{2}{x+1}\right)} dx}$	$= \frac{\pi}{2} \approx 1.5708$	1.6085019683186124	2.076910604996239	1.609379826084149	$E_{KGL2} = 2.3553490365081 \times 10^{-2}$ $E_{NDO} = 2.60795542073356 \times 10^{-1}$ $E_{OHR} = 2.30205838669871 \times 10^{-2}$

Table: (2). Numerical results computed by the closed-type hybrid quadrature rule $I_{CHR}(f)$ compared with its constituent rules $I_{KGL2}(f)$ rules and $I_{NDC}(f)$.

Integral	Near-Exact	$I_{KGL2}(f)$	$I_{NDC}(f)$	$I_{CHR}(f)$	Relative Error
$I_1 = \int_1^2 e^{e^x} dx$	≈ 255.6758679186	255.820070562722	253.89134362084	255.81645588145	$E_{KGL2} = 5.6400568941 \times 10^{-4}$ $E_{NDC} = 6.9796352399 \times 10^{-3}$ $E_{CHR} = 5.4986794034 \times 10^{-4}$
$I_2 = \int_1^2 e^{-x^2} dx$	≈ 0.1352572579499946	0.13525734814014	0.13525655014006	0.1352573466445	$E_{KGL2} = 6.6680450970 \times 10^{-7}$ $E_{NDC} = 5.2330623751 \times 10^{-6}$ $E_{CHR} = 6.55747403635 \times 10^{-7}$
$I_3 = \int_1^2 \frac{\sin x}{x} dx$	≈ 0.6593299064355118	0.6593299064397252	0.6593299064006406	0.659329906439652	$E_{KGL2} = 6.3904387767 \times 10^{-12}$ $E_{NDC} = 5.288887503 \times 10^{-11}$ $E_{CHR} = 6.2793036395 \times 10^{-12}$
$I_4 = \int_1^2 \frac{e^{-x}}{x} dx$	≈ 0.170483	0.17048364153294074	0.1704813324558944	0.17048363720543466	$E_{KGL2} = 1.27781034023 \times 10^{-6}$ $E_{NDC} = 1.226648034 \times 10^{-5}$ E_{CHR}=$1.2524266050 \times 10^{-6}$
$I_5 = \int_0^1 \frac{dx}{1+x^4}$	≈ 0.86697	0.8669767626543958	0.8669225103807843	0.8669766609786879	$E_{KGL2} = 4.3545929803 \times 10^{-6}$ $E_{NDC} = 5.8222066735 \times 10^{-5}$ $E_{CHR} = 4.23731630684 \times 10^{-6}$

Table: (3). Numerical results computed by the closed-type hybrid quadrature rule $I_{CBoLob}(f)$ compared with its constituent rules $I_{RBool}(f)$ rules and I_{NLob}.

Integral	$I_{RBool}(f)$	$I_{NLob}(f)$	I_{CBoLob}	Relative Error
$\int_1^2 e^{-x^2}\,dx$	0.1352600678077	0.1352461059959	0.135246113746	$E_{RBool} = 2.0774173353466 \times 10^{-5}$ $E_{NLob} = 8.2449949905523 \times 10^{-5}$ $E_{CBoLob} = 8.239264762395 \times 10^{-5}$
$\int_1^2 \frac{\sin x}{x}\,dx$	0.659329912197	0.6593298963945	0.659329900299	$E_{RBool} = 8.739114546289 \times 10^{-9}$ $E_{NLob} = 8.2449949905523 \times 10^{-5}$ $E_{CBoLob} = 8.2392647623949 \times 10^{-5}$
$\int_1^2 \frac{e^{-x}}{x}\,dx$	0.170475250379	0.1705180434985	0.1705180554623	$E_{RBool} = 4.7941954266458 \times 10^{-5}$ $E_{NLob} = 2.03068488283273 \times 10^{-4}$ $E_{CBoLob} = 2.0313866405311 \times 10^{-4}$

Table: (4). Numerical results computed by the Closed-type hybrid quadrature rule $I_{CHR}(f)$ compared with its constituent rules and $I_{KGL2}(f)$ rules and $I_{NDC}(f)$ using an adaptive algorithm with termination criterion 10^{-5}.

Integral	Iterations $I_{KGL2}(f)$	Iterations	Iterations $I_{NDC}(f)$	Iterations	Iterations $I_{CHR}(f)$	Iterations
$I_1 = \int_1^2 e^{e^x}\,dx$	255.675894812	2	255.667210454	12	255.675608551	10
$I_2 = \int_1^2 e^{-x^2}\,dx$	0.135257258	1	0.135249563462	6	0.135254562	1
$I_3 = \int_1^2 \frac{\sin x}{x}\,dx$	0.659329906	1	0.659336124	6	0.659330466	1
$I_4 = \int_1^2 \frac{e^{-x}}{x}\,dx$	0.170480601	1	0.170479739	9	0.170480601391	1

Table: (5). Numerical results computed by the open-type hybrid quadrature rule $I_{OHR}(f)$ compared with its constituent rules $I_{KGL2}(f)$ rules and $I_{NDO}(f)$ using adaptive algorithm.

Integral	$I_{KGL2}(f)$	Iterations	$I_{NDO}(f)$	Iterations	$I_{CHR}(f)$	Iterations
$I_1 = \int_1^2 \ln[\ln(x)]\,dx$	−1.2009667766545051	10	−1.200967848901904	12	−1.20096674432545	10
$I_2 = \int_0^1 \sqrt{1-x^4}\,dx$	0.8740193510435356	7	0.8740193218057818	9	0.8740193527868432	7
$I_5 = \int_0^\infty \frac{e^{-x}}{1+x^2}\,dx$	0.621449500800718	1	0.6214593348845058	12	0.6214519128864372	5
$I_7 = \int_0^\infty \frac{\sin x}{x}\,dx$	1.5888888194648858	13	1.5689338860146185	10	1.5886295131152703	13

CONCLUSIONS

Open and closed hybrid quadrature rules $I_{OHR}(f)$, $I_{CHR}(f)$, and $I_{CBoLob}(f)$ have been proposed in this paper. Their ingredients are some enhanced quadrature rules, such as the Kronrod-Legendre pair, Bool's rule enhanced by Richardson extrapolation, the numerically enhanced Gauss-Lobatto quadrature rule, and derivative-based Newton-Cotes formulae. The proposed hybrid quadrature rules are found to perform better than their ingredient quadrature rules through error analysis, as evidenced in equations (9), (12), and (19). Strictly speaking, the degree of precision of the proposed hybrid quadrature rules is $(p+2)$, where p is the degree of precision of its ingredient rules. A variety of integral examples have been considered for verification purposes that correspond to numerous applications in science and engineering. Considering that we implement low-order quadrature rules of either the Gauss or Newton-Cotes type, the observed accuracy is satisfied. The performance of such hybrid rules can be enhanced by the adaptive quadrature rule as shown in Tables (4) and (5) with a tolerance of 10^{-5}. Overall, all the results obtained are very satisfactory.

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