Al-Mukhtar Journal of Sciences 36 (1): 42-56, 2021

α-Reflexive Rings with Involution

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Received: 13 January 2021/ Accepted: 31 January 2021
Doi: https://doi.org/10.54172/mjsc.v36i1.22

Abstract: This paper studies the concept of the α-quasi-*-IFP (resp., α -*-reflexive) *-rings, as a generalization of the quasi-*-IFP (resp., *-reflexive) *-rings and every quasi-*-IFP (resp., *-reflexive) *-ring is α -quasi-*-IFP (resp., α-*-reflexive). This paper also discusses the sufficient condition for the quasi-*-IFP (resp., *-reflexive) *-ring in order to be α-quasi-*-IFP (resp., α-*- reflexive). Finally, this study investigates the α-quasi-*-IFP (resp., α-*-reflexivity) by using some types of the polynomial rings.

Keywords: **-reduced; *-rigid; α-*-rigid; α-*-IFP; α-quasi-*-IFP; α-*-reversible; α-*-reflexive *-rings.*

INTRODUCTION

Throughout this paper, R denotes an associative *-ring with unity and α denotes a nonzero nonidentity endomorphism of a given *-ring, unless specified otherwise. IFP stands for "insertion-of-factors property", R is semicommutative or has IFP if the right annihilator $r(a) = \{x \in R | ax = 0\}$ of every element $a \in R$ is a two-sided ideal. A *-ring R is said to have IFP when all $ab \in R, ab = 0$ which implies that $aRb = 0$ by (Kim & Lee, 2003). In both studies (Başer & Kwak, 2010) and (Başer et al., 2008) discussed an endomorphism α of a ring R, the endomorphism α is called semicommutative if $ab = 0$ implies $aR\alpha(b) = 0$ for $a \in R$. Also, a ring R is called α- semicommutative, if there exists a semicommutative endomorphism α of R.

Another study (Zhao & Zhu, 2012) shows that, an endomorphism α of a ring R is called reflexive whenever $aRb = 0$ for $a, b \in R$, $bR\alpha(a) = 0$. A ring R is called α-reflexive if there exists a reflexive endomorphism α of R.
A *-ring R is said to have *-IFP if all $a, b \in R, ab = 0$ implies $aRb^* = 0$. For more details see (Aburawash & Saad, 2014). By (Aburawash & Saad, 2019) R has quasi-*-IFP if all $a, b \in R, ab = ab^* = 0$ implies $aRb = 0$, a *-ring R is called *-reversible (resp., *-reflexive) if for all $a, b \in R, ab = ab^* = 0$ (resp., $aRb = aRb^* = 0$) implies $ba = 0$ (resp., $bRa = 0$).

According to (Abdulhafed, 2019), a *-ring R is said to be *-rigid if for $a, b \in R, ab^2 = abb^* = 0$ implies $ab = 0$, an α be a *-endomorphism of R. α is called a *-rigid *-endomorphism if $a\alpha(a) = a\alpha(a^*) = 0$ implies $a = 0$ for all $a \in R$. A *-ring R is called α-*-rigid if there exists a *-rigid *-endomorphism α of R and a *-endomorphism α of a *-ring R is called *-reversible if whenever $ab = ab^* = 0$, then $\alpha(b)\alpha(a) = 0$, for $a, b \in R$ (also, $\alpha(b^*)\alpha(a) = 0$). A *-ring R is called α-*-reversible if there exists a *-endomorphism α on R. A *-rigid *-rings are equivalent to *-reduced *-rings.

In view of the studies mentioned above, this paper introduces the class of α-quasi-*-IFP (resp., α-*-reflexive) *-rings, which is the *-

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version (and also a generalization) of the quasi-*-IFP (resp., *-reflexive) *-rings.

Moreover, some properties and results of these classes of *-rings are investigated. The class of α-*-reflexive *-rings is introduced as a generalization of *-reflexive and *-reduced *-rings, since, by definition, *-reflexive *-rings are α-*-reflexive *-rings and α-*-reversible is α-*-reflexive.

Also, other relative results are given. Here, finally, we conclude the results of the paper by explaining the diagram and the relations among the corresponding classes.

α-*-IFP rings

In this section, α-*-IFP *-rings are introduced as a generalization of *-IFP *-rings.

Definition 1. A *-endomorphism α of a *-ring R is called **-IFP* if whenever $ab = 0$, then $\alpha(arb^*) = 0$ for all $a, b, r \in R$. A *-ring R is called *α-*-IFP* if there exists a *-endomorphism α on R.

It is clear that a ring R is *-IFP if R is I_R-*-IFP, where I_R is the identity *-endomorphism of R. It is easy to see that every *-subring S with $\alpha(S) \subseteq S$ of an α-*-IFP *-ring is also α -*-IFP.

Obviously, in general, the reverse implication in the above definition does not hold by the following example which also shows that there exists *-endomorphism α of α -*-IFP *-ring R such that R is not α -IFP.

Example 1. Assume both $\mathbb{F}$ to be a field, the *-ring $R = \mathbb{F} \oplus \mathbb{F}$ with exchange involution $(a, b)^* = (a^*, b^*)$ and *-endomorphism $\alpha: R \to R$ is given by $\alpha((a, b)) = (b, a)$ for all $a, b \in \mathbb{F}$. Since Since A=(1,0), B=(0,1),$A = (1,0\ \ ), B = (0,1)R$ clearly α-*-IFP, but it does not have α-IFP.

Proposition 1. Let R be α- *-IFP *-ring and α is *-monomorphism on R, then R *-IFP.

α-quasi-*-IFP rings

In this part of the paper the focus is on the α-quasi-*-IFP *-rings and how to introduce a generalization for quasi-*-IFP *-rings.

Definition 2. A *-endomorphism α of a *-ring R is called *quasi-*-IFP,* when $ab = 0 = ab^*$, then $\alpha(arb) = 0$, for all $a, b, r \in R$ (consequently $\alpha(arb^*) = 0$). A *-ring R is called *α-quasi-*-IFP* if there exists a *-endomorphism α on R.

It is clear that it is needed to exclude the identity *-endomorphism I_R, because the *-ring R is I_R-quasi-*-IFP if and only if R is quasi-*-IFP. In general, α-quasi-*IFP *-ring R is quasi-*-IFP if α is a *-monomorphism on R.

Proposition 2. *Let* R *be* α*-quasi-*-IFP *-ring and* α *is *-monomorphism on* R*, then* R *quasi-*-IFP.*

Proposition 3. *Let* R *be* α *is *-monomorphism a *-ring. If* R *is* α*-quasi-*-IFP and it has *-IFP, then* R *is IFP.*

Proof. It is obvious, since $ab = 0$, implies $aRb^* = 0$, by the *-IFP property and R α -quasi-*-IFP, we have $\alpha(arb) = 0$, for all $a, b, r \in R$. Hence, $aRb = 0$ since α is *-monomorphism. Thus R is IFP.

It is clear that, a ring R is quasi-*-IFP if $R$is I_R-quasi-*-IFP, where I_R is the identity *-endomorphism of R. It is easy to note that every *-subring S with $\alpha(S) \subseteq S$ of an α-quasi-*-IFP *-ring is also α-quasi-*-IFP.

Notice that, in general, the reverse implication in the above definition does not hold by the following example, which shows also that, there exists *-endomorphism α of α-quasi-*-IFP *-ring R such that R is not α-IFP.

ISSN: online 2617-2186 print 2617-2178

Example 2. The *-ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, with the changeless involution * defined as $(a,b)^* = (a^*, b^*)$ and *-automorphism $\alpha: R \to R$ given by $\alpha\big((a,b)\big) = (b,a)$ is not α-IFP but, it is IFP, since the nonzero element $A(0,1), B = (1,0)$ satisfies that $AB = AB^* = 0$, while $\alpha(A)RB \neq 0$ and also $AR\alpha(B) \neq 0$, $ARB = 0$. Moreover, R is α-quasi-*-IFP.

Proposition 4. *Let* R *be an* α-*-reversible *-*ring, then* R *is* α-*quasi-*-IFP.*

The converse to **Proposition 4** is not true according on the following example:

Example 3. The *-ring $R = \begin{pmatrix} \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{F} \end{pmatrix}$ over a field $\mathbb{F}$, with the adjoint involution * is α-quasi-*-IFP. Moreover, R with the *-endomorphism $\alpha: R \to R$ defined by $\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}$, is not α-*-reversible by (Abdulhafed, 2019). The following example declares that, $T_4(R)$ is not an $\tilde{\alpha}$-quasi-*-IFP *-ring, even if R is an α-rigid *-ring. Since *-endomorphism α of a *-ring R is also extended to the *-endomorphism $\tilde{\alpha}$ of $T_4(R)$ defined by $\tilde{\alpha}\left((a_{ij})\right) = (\alpha(a_{ij}))$.

Example 4. Consider R be a commutative α-rigid *-ring. Then, the *-ring $T_4(R)$ is not $\tilde{\alpha}$-quasi-*-IFP, since the matrices

$$A = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in T_4(R),$$

satisfy $AB = AB^* = 0$ while, $\tilde{\alpha}(A)\tilde{\alpha}(C)\tilde{\alpha}(B) \neq 0$, while, $\tilde{\alpha}(A)\tilde{\alpha}(C)\tilde{\alpha}(B) \neq 0$ for

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in T_4(R),$$

where $\alpha(e) = e$, by (Abdulhafed, 2019). Thus $\tilde{\alpha}(A)\tilde{\alpha}(T_4(R))\tilde{\alpha}(B) \neq 0$ and so $T_4(R)$is not $\tilde{\alpha}$-quasi-*-IFP. Similarly, it can be proved that $T_n(R)$ is not $\tilde{\alpha}$-quasi-*-IFP for $n \geq 5$.

Furthermore, the class of the α-quasi-*-IFP *-rings is closed under the finite direct sums (with changeless involution). In addition to assume R be a *-ring. Then, both eR and $(1-e)R$ are α-quasi-*-IFP for some projection e in R with $\alpha(e) = e$, if and only if R is α-quasi-*-IFP.

α-reflexive rings with involution

In this section, α-*-reflexive *-rings are introduced as a generalization for *-reflexive and *-rigid *-rings.

Definition 3. A *-endomorphism α of a *-ring R is called **-reflexive* when $aRb = aRb^* = 0$, then, $\alpha(bra) = 0$, for all $a, b, r \in R$ (consequently $\alpha(b^*ra) = 0$). A *-ring R is called *α-*-reflexive,* if there exists a *-endomorphism α on R.

Every *-reflexive *-ring is clearly α-*-reflexive, but, on the opposite side it is not true as shown by the following example.

Example 5. The *-ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$, with the adjoint involution * and *-endomorphism $\alpha: R \to R$ is defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$$

is α-*-reflexive, since if the matrices

$$A = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}, B = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} \in R,$$

satisfy $ARB = ARB^* = 0$, then, we get the equations: $a_1 a a_2 = a_1 a c_2 = 0, a_1 a b_2 +$

ISSN: online 2617-2186 print 2617-2178

$a_1bc_2 + b_1cc_2 = -a_1ab_2 + a_1ba_2 + b_1ca_2 = 0$ and $c_1cc_2 = c_1ca_2 = 0$, which implies that:

$$\alpha(BrA) = \begin{pmatrix} a_2aa_1 & 0 \\ 0 & c_2cc_1 \end{pmatrix} = 0.$$

Moreover, R is not *-reflexive, since

$$BRA = \begin{pmatrix} 0 & a_2bc_1 + b_2cc_1 \\ 0 & 0 \end{pmatrix} \neq 0.$$

Next, here it is also deduced that, it excluded the identity *-endomorphism I_R, because the *-ring R is I_R-*-reflexive if and only if R is *-reflexive. In general, α-*-reflexive *-ring R is *-reflexive if α is a *-monomorphism on R.

Proposition 5. *Let* R *be an* α-**-reflexive *-ring and* α *is *-monomorphism on* R, *then* R *is *-reflexive.* It is clearly visible that, there is no connection between either α- reflexive or α-*-rigid and α-*- reflexive *-rings. According to the example 5, there exists an α-*-reflexive *-ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ which is not α- reflexive. Since

$$BR\alpha(A) = \begin{pmatrix} a_2aa_1 & a_2ab_1 + a_2bc_1 \\ 0 & c_2cc_1 \end{pmatrix} \neq 0,$$

and

$$BR\alpha(A) = \begin{pmatrix} a_2aa_1 & a_2bc_1 + a_2cc_1 \\ 0 & c_2cc_1 \end{pmatrix} \neq 0.$$

Example 6. Consider $R = \mathbb{Z}_8 \oplus \mathbb{Z}_8$ with usual addition and multiplication with exchange involution $(a,b)^* = (a^*, b^*)$, and let $\alpha : R \to R$ be an *-endomorphism is defined by $\alpha((a,b)) = (b,a)$. For $a = (4,2), b(2,0) \in \mathbb{Z}_8 \oplus \mathbb{Z}_8$, we get $aRb = aRb^* = 0$.

However, $b\alpha(a) = (4,0) \neq 0$ and $\alpha(b)a = (0,4) \neq 0$, entailing neither $bR\alpha(a) = 0$ and nor $\alpha(b)Ra = 0$. Hence, R is neither right nor left α- reflexive, but, it is α-*-reflexive.

Proposition 6. *Let* R *be a *-reflexive *-ring. Then the following are equivalent.*

1. R *is* α-**- reflexive.*
2. *If* $arb = 0 = arb^*$ *for all* $a, b, r \in R$, *then* $\alpha(arb) = \alpha(arb^*) = 0$.

Proof. 1 ⇒ 2. Let $arb = arb^* = 0$, where $a, b, r \in R$, then $\alpha(arb) = \alpha(arb^*) = 0$. Hence, $\alpha(b)\alpha(r)\alpha(a) = 0 = \alpha(b^*)\alpha(r)\alpha(a)$ for all $r \in R$, since R is *-reflexive, then, we get that $\alpha(a)\alpha(r)\alpha(b) = 0 = \alpha(a)\alpha(r)\alpha(b^*)$ which implies $\alpha(aRb) = \alpha(aRb^*)$.

2 ⇒ 1. Let $arb = 0 = arb^*$ for $a, b, r \in R$ which implies by **2** that $\alpha(arb) = \alpha(arb^*) = 0$. Since R is *- reflexive, then $\alpha(bRa) = 0$.

Proposition 7. *Let* α *be a *-monomorphism on a *-ring* R. *Then,* R *is an* α-**- reflexive *-ring if and only* $\alpha(a)\alpha(r)\alpha(b) = 0 = \alpha(a)\alpha(r)\alpha(b^*)$ *if implies* $bra = 0 = b^*ra$ *for all* $a, b, r \in R$.

Proof. Let $\alpha(a)\alpha(r)\alpha(b) = 0 = \alpha(a)\alpha(r)\alpha(b^*)$ for $a, b, r \in R$, then

$\alpha(\alpha(b))\alpha(\alpha(r))\alpha(\alpha(a)) = \alpha^2(bra) = 0 = \alpha(\alpha(b^*))\alpha(\alpha(r))\alpha(\alpha(a)) = \alpha^2(b^*ra)$Since, R is α-*-reflexive and α is a *-monomorphism imply $bRa = 0 = b^*Ra$.

Conversely, let $arb = arb^* = 0$, where $a, b, r \in R$, then, $\alpha(arb) = \alpha(a)\alpha(r)\alpha(b) = 0 = \alpha(a)\alpha(r)\alpha(b^*) = \alpha(arb^*)$ by hypothesis so $bRa = 0 = b^*Ra$.

It is easily to show that the class of α-*- reflexive *-rings is closed under finite direct sums (with changeless involution).

Proposition 8. *The class of* α-**- reflexive *-rings is closed under finite direct sums.*

The next step, an example is needed to explain that *-reflexivity is not closed under taking *-subrings. The full matrix ring $\mathbb{M}_n(R)$ over a *-ring R with adjoint involution and the *-endomorphism $\alpha: \mathbb{M}_n(R) \to \mathbb{M}_n(R)$ is defined by

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$$\alpha\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix},$$

It is not α-*-reflexive for $n \geq 2$, according to the following example:

Example 7. The ring $R = \mathbb{M}(\mathbb{Z}_2)$ is prime and *-reflexive. The upper triangular matrix ring $T_2(R) = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ over $\mathbb{Z}_2$ is a *-subring of R. R is clearly α-*-reflexive, but $T_2(R)$ is not, since both matrices $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ of R satisfy

$ARB = ARB^* = 0$, but $\alpha(B)R\alpha(A) =$

$$\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \neq 0$$

According to the study by [(Abdulhafed, 2019)] every *-reduced equivalence *-rigid, [(Aburawash & Saad, 2016), Example 4.2 and Proposition 4.6] and [(Aburawash & Saad, 2019), Example 7 and Corollary 3], we deduce the following important results.

Corollary 1. Every *-rigid *-ring is α-*- reflexive.

The opposite of the previous **Corollary 1** is not true since, by **Examples 5** it is obtained α-*-reflexive and it is not *-reduced.

Corollary 2. Every *-Baer *-ring is α-*- reflexive.

Corollary 3. *Every *-domain is* α-*- *reflexive.*

Corollary 4. *Every *-ring with semiproper involution is* α-*- *reflexive.*

Now, in contrast to the previous corollary, it is not necessary to be true as shown in the following example.

Example 8. From [(Aburawash & Saad, 2019), Example 9], if $\mathbb{F}$ is a field, then the ring $R = \mathbb{F} \oplus \mathbb{F}^{OP}$, with the exchange involution * is defined by $(a, b)^* = (b, a)$ and the *-endomorphism $\alpha = *$ for all $a, b \in R$, it is obvious that an α *-reflexive but, * is not semiproper. Indeed, the element $0 \neq A = (0, a)$ for some nonzero element a of $\mathbb{F}$ satisfy $ARA^* = 0$.

The following proposition and example show that the class of α-*-reflexive *-rings generalizes strictly that of α-*-reversible *-rings.

Proposition 9. *Every α-*-reversible *-ring is α-*- reflexive.*

Proof. Let $aRb = aRb^* = 0$, then, $ab = ab^* = 0$ implies $rab = rab^* = 0$ for every $r \in R$. So that $\alpha(bra) = \alpha(b^*ra) = 0$, from the α-*-reversibility of R. Thus $\alpha(bRa) = \alpha(b^*Ra) = 0$, hence, R is α-*- reflexive.

The question that, when a α-*-reflexive *-ring is α-*-reversible is answered by the following proposition.

Proposition 10. *A *-ring R is α-*-reversible if and only if R has quasi-*-IFP and α-*- reflexive.*

Proof. The necessity is clear to sufficiency, let's consider $ab = ab^* = 0$, for some $a, b \in R$. Since R has quasi-*-IFP, then, $aRb = aRb^* = 0$. The α-*-reflexivity of R implies $aRb = aRb^* = 0$. Hence $\alpha(ba) = \alpha(b^*a) = 0$, and R is α-*-reversible. By the **Corollary 1**, we can get the following result.

Corollary 5. *Every α-rigid *-ring is α-*- reflexive.*

The following example can show that the converse of **Corollary 5** is not true.

Example 9. By looking to the study [(Başer et al.,2009), Example 2.7 (i)], the trivial extension *-ring $T(\mathbb{Z}_4, \mathbb{Z}_4)$, with the adjoint involution * and the *-endomorphism $\alpha = *$ is not semiprime (so not α-*-rigid), but it is α-*- reflexive.

ISSN: online 2617-2186 print 2617-2178

Proposition 11. *Let's consider* α *to be a *-monomorphism on a *-ring* R*, the following statements are equivalent:*

i. *R is α-*- reflexive.*
ii. $r_*(aR) = l_*(Ra)$ *for every element* $a \in R$.
iii. *For any two nonempty subsets A and B of R,* $ARB = ARB^*$ *implies* $\alpha(BRA) = 0$ *(consequently* $\alpha(B^*RA) = 0$).

Proof. $(i) \Rightarrow (ii)$. Let $x \in r_*(aR)$, then, $aRx = aRx^* = 0$. Since R is α-*- reflexive, we have $\alpha(x)\alpha(Ra) = \alpha(x^*)\alpha(Ra) = 0$, but α is a *-monomorphism, so $xRa = x^*Ra = 0$, for every $a \in R$. Hence, $xRa = x^*Ra = 0$ implies $x \in l_*(Ra)$, and we get $r_*(aR) \subseteq l_*(Ra)$. Similarly, $l_*(Ra) \subseteq r_*(aR)$ and so $r_*(aR) = l_*(Ra)$ follows.

$(ii) \Rightarrow (iii)$. Let $ARB = ARB^*$ for some subsets A and B of R. Then $B \subseteq r_*(AR)$ and so $aRb = aRb^* = 0$ for all $a \in A$ and $b \in B$ and hence $b \in r_*(aR) = l_*(Ra)$ and $bRa = b^*Ra = 0$. which implies $\alpha(BRA) = \alpha(B^*RA) = 0$.

$(iii) \Rightarrow (iv)$. is obvious.

From **Proposition 11**, we have the following corollary.

Corollary 6. *Let α be a *-monomorphism of a *-ring*R*, then the following statements are equivalent:*

i. *R is *- reflexive.*
ii. *R is α-*- reflexive.*
iii. $r_*(aR) = l_*(Ra)$ *for every element* $a \in R$.
iv. *For any two nonempty subsets A and B of R,* $ARB = ARB^* = 0$ *implies* $\alpha(BRA) = 0$ *(consequently* $\alpha(B^*RA) = 0$*).*

Again, a *-domain *-ring is *- reflexive [(Aburawash & Saad, 2019), Example 4], then, we have:

Corollary 7. *Every *-domain *-ring is α-*- reflexive.*

The converse of **Corollary 7** is not true, since $T(\mathbb{Z}_4, \mathbb{Z}_4)$ is not a domain *-ring in the **Example 9**

Proposition 12. *Let's assume* α *be *-endomorphisms of* a **-ring* R*. If* R *is* α-*- *reflexive *-ring, then* $aRb = 0 = aRb^*$ *for* $a, b \in R$ *implies* $\alpha^k(a)R\alpha^k(b) = 0 = \alpha^k(a)R\alpha^k(b^*)$ *and* $\alpha^k(b)R\alpha^k(a) = 0 = \alpha^k(b^*)R\alpha^k(a)$*for all* $k \geq 1$.

For *-endomorphism α and projection e of a *-ring R such $\alpha(e) = e$, that, we have *-endomorphism $\tilde{\alpha}: eRe \to eRe$ is defined by $\tilde{\alpha}(ere) = e\alpha(r)e$, one can show that *- reflexive property is extended to the *-corner.

Proposition 13. *Let* R *be* a α-*- *reflexive *-ring, then, the *-corner* eRe *for every projection e of* R *is also* α-*- *reflexive.*

Proof. Let R be α-*-reflexive and $a = exe$, $b = eye \in eRe$ such that $a(eRe)b = a(eRe)b^* = 0$. Then $exeReye = exeRey^*e = 0$ implies $\tilde{\alpha}(eye)R\tilde{\alpha}(exe) = \tilde{\alpha}(ey^*e)R\tilde{\alpha}(exe) = 0$, since R is α-*- reflexive. Therefore $\alpha(b)(eRe)\alpha(a) = \alpha(b^*)(eRe)\alpha(a) = 0$ and so eRe is α-*- reflexive.

Proposition 14. *Let* R *be* a **-ring with *-endomorphism* α *such that* $\alpha(e) = e$ *for* $e^2 = ee^* = e \in R$*. If* e *is a central projection* R*, then,* eR *and* $(1 - e)R$ *are* α-*- *reflexive if and only if* R *is* α-*- *reflexive.*

Proof. It is enough to show the necessity by **Proposition 8**. Suppose that eR and $(1 - e)R$ are α-*-reflexive for a central projection $e \in R$. Let $aRb = 0 = aRb^*$ for $a, b \in R$. Then,

$$ea(eR)eb = 0 = ea(eR)eb^*,$$

and

$$(1 - e)a(1 - e)R(1 - e)b = 0 = (1 - e)a(1 - e)R(1 - e)b^*.$$

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By hypothesis,

$$0 = \tilde{\alpha}(eb)eR\tilde{\alpha}(ea) =$$

$$e\alpha(b)e\alpha(a) = e\alpha(b)R\alpha(a)$$

and

$$0 = \tilde{\alpha}\big((1-e)b\big)(1-e)R\tilde{\alpha}\big((1-e)a\big) = (1-e)\alpha(b)(1-e)R(1-e)\alpha(a) = (1-e)\alpha(b)R\alpha(a) = 0.$$

For a ring R and an endomorphism, $\alpha: R \to R$ the skew polynomial ring (also called on Ore extension of endomorphism type) $R[x;\alpha]$ of R is the ring that is obtained by giving the polynomial ring over R with the new multiplication $xr = \alpha(r)x$ for all $r \in R$.

Proposition 15. *If* R *and* $R[x;\alpha]$ *(resp.,* $R[[x;\alpha]]$*) are *- reflexive, then,* R *is* α*-*-rigid.*

Proof. Let us consider $a\alpha(a) = a\alpha(a^*) = 0$ and $p = ax, q = a \in R[x;\alpha]$ for $a \in R$, then

$pRq = axa = a\alpha(a)x = 0, pRq^* = axa^* = a\alpha(a^*)x = 0$. Since $R[x;\alpha]$ is *- reflexive, hence $qRp = aax = a^2x = 0$, $q^*Rp = a^*ax = a^*ax = 0$ and so $a^2 = a^*a = 0$. Since R is *-reduced, we get $a = 0$.

According to (Abdulhafed, 2019) and **Proposition 15** the following results are straightforward:

Corollary 8. *If both* R *and* $R[x;\alpha]$ *(resp.,* $R\big[[x;\alpha]\big]$*) are a reflexive *-ring, then,* R *is* α*-*-rigid.*

Corollary 9. *If* R *is an* α*-*-rigid *-ring and* $R[x;\alpha]$ *(resp.,* $R\big[[x;\alpha]\big]$*) is a reversible *-ring, therefore,* R *is *-reflexive.*

Corollary 10. *If* R *is an* α*-*-rigid *-ring, then,* $R[x;\alpha]$ *(resp.,* $R\big[[x;\alpha]\big]$*) is *-reflexive.*

If the *-ring $R[x;\alpha]$ is *-reflexive, then, R is *-reflexive and the converse is not correct. Also, the converse of **Corollary 9** is wrong according to studying the example in (Abdulhafed, 2019).

Next, the α-*-reflexivity and α-quasi-*-IFP do not imply each other.

Example 10. For a commutative *-ring R,

$$T_3(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| a,b,c,d \in R \right\},$$

it has quasi-*-IFP, but, it does not have *-reflexive by [(Aburawash & Abdulhafed, 2018a), Example 5]. Thus, R is id_R-quasi-*-IFP, but it is not id_R-*-reflexive.

As, a consequence from **Example 10**, $T_n(R)$ is not α-*-reflexive for $n \geq 4$. Clearly, if R is a commutative *-ring, hence, the *-ring:

$$T_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & a \end{pmatrix} \middle| a, a_{ij} \in R, n \geq 3 \right\}$$

, it is not α-*-reflexive according to (Abdulhafed, 2019) and is quasi-*-IFP. However, it is clearly evident that $T_4(R)$ is not α -quasi-*-IFP and so $T_n(R)$ is not quasi-*-IFP for $n \geq 4$ as reported in **Example 4**.

We get the same results as stated in both studies **[**(Aburawash & Abdulhafed, 2018a), Corollary 11(2)**]** and (Abdulhafed, 2019).

Corollary 11. *If* R *is a semiprime *-ring, then, the trivial extension* $T(R,R)$*, with adjoint involution is* α*-*-reflexive.*

Corollary 12. *Let's assume* R *to be a reduced *-ring and* α *is the *- endomorphism on* R*, then, the *-ring* $T(R,R)$*, with componentwise involution * is* $\tilde{\alpha}$*-*-reflexive.*

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Corollary 13. *Suppose R be an α-rigid *-ring,* then*, the *-ring $T(R,R)$, with componentwise involution* * is *$\tilde{\alpha}$-*-reflexive.*

Corollary 14. Let R be *a rigid *-ring,* then*, the *-ring $T(R,R)$, with componentwise involution* * is *$\tilde{\alpha}$-*-reflexive.*

The trivial extension $T(R,R)$ of a *-ring R can be extended to a *-ring

$$T_3(R) = \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix},$$

and *-endomorphism α of a *-ring R, it also is extended to the *-endomorphism $\tilde{\alpha}: T_3(R) \to T_3(R)$ is defined by $\tilde{\alpha}(a_{ij}) = \left(\alpha(a_{ij})\right)$ with involution defined as:

$$\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}^* = \begin{pmatrix} a & d & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}$$

The following example shows that $T_3(R)$ cannot be $\tilde{\alpha}$-*-reflexive even if R is α-rigid-*-ring.

Example 11. Let R be a commutative α-rigid *-ring. Then, the *-ring $T_3(R)$ is not $\tilde{\alpha}$-*-reflexive, since the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and}$$

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R),$$

Satisfy

$$ABC = ABC^* = 0,$$

while,

$$\tilde{\alpha}(C)\tilde{\alpha}(B)\tilde{\alpha}(A) \neq 0,$$

where $\alpha(e) = e$, according to (Abdulhafed, 2019).

Extensions of α-quasi-*-IFP and α-*-reflexive *-rings

In this section, recall that R be a *-ring and S be a multiplicatively closed subset of R consisting of nonzero central regular elements, then, the localization of R to S is $S^{-1}R = \{u^{-1}a \mid u \in S, a \in R\}$, and it also is a *-ring with involution $\diamond$ defined as: $(u^{-1}a)^\diamond = u^{-1^*}a^* = u^{*^{-1}}a^*$ see for more details (Aburawash & Abdulhafed, 2018b). A *-endomorphism $\tilde{\alpha}$ on R can be extended to $\tilde{\alpha}$ on $S^{-1}R$, the mapping $\tilde{\alpha}: S^{-1}R \to S^{-1}R$ is defined by $\tilde{\alpha}(u^{-1}a) = \alpha(u^{-1})\alpha(a)$ see [(Abdulhafed, 2019)]. Then, the following proposition is obtained.

Proposition 16. *A *-ring R is α-quasi-*-IFP if and only if $S^{-1}R$* ***is*** *$\tilde{\alpha}$-quasi-*-IFP.*

Proof. Assume that $\beta\gamma = 0 = \beta\gamma^\diamond$ with $\beta = u^{-1}a$ and $\gamma = v^{-1}b$, where $a, b \in R$ and $u, v \in S$. Hence,

$$\beta\gamma = u^{-1}av^{-1}b = u^{-1}v^{-1}ab = (vu)^{-1}ab = 0$$

and

$$\beta\gamma^\diamond = u^{-1}a(v^*)^{-1}b^* = u^{-1}(v^*)^{-1}ab^*$$

$$= (v^*u)^{-1}ab^* = 0,$$

since S exists in the center of R, and so $ab = ab^* = 0$. By hypothesis $acb = 0$ for all $c \in R$, which implies

$$\tilde{\alpha}(\beta\ \xi\ \gamma) = \tilde{\alpha}\left(u^{-1}aw^{-1}cv^{-1}b\right) = \alpha\left((vwu)^{-1}\right)\alpha(acb) = 0$$

for every $\xi = w^{-1}c \in S^{-1}R$. The converse is clear.

Proposition 17. *A *-ring R is α -*-reflexive if and only if $S^{-1}R$ is $\tilde{\alpha}$-*-reflexive.*

Proof. It is enough to show that $S^{-1}R$ is $\tilde{\alpha}$ -*-reflexive if R is α *-reflexive. Let R be an α-*-reflexive and $\beta\xi\gamma = 0 = \beta\xi\gamma^\diamond$ with

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$\beta = u^{-1}a,\ \xi = w^{-1}c$ and $\gamma = v^{-1}b$, where $a, b, c \in R$ and $u, v, w \in S$. Hence,

$\beta\xi\gamma = u^{-1}aw^{-1}cv^{-1}b = u^{-1}w^{-1}v^{-1}acb = (vwu)^{-1}acb = 0,$

and

$\beta\xi\gamma^{\diamond} = u^{-1}aw^{-1}c(v^{*})^{-1}b^{*} = u^{-1}w^{-1}(v^{*})^{-1}acb^{*} = (v^{*}wu)^{-1}acb^{*} = 0.$

Since S exists in the central of R so

$acb = 0 = acb^{*}.$

By hypothesis $\alpha(b)\alpha(c)\alpha(a) = 0$, which implies that:

$\tilde{\alpha}(\gamma)\tilde{\alpha}(\xi)\tilde{\alpha}(\beta) = \tilde{\alpha}(v^{-1}b)\tilde{\alpha}(w^{-1}c)\tilde{\alpha}(u^{-1}a)$
$= \alpha(v^{-1})\alpha(w^{-1})\alpha(u^{-1})\alpha(b)\alpha(c)\alpha(a)$
$= \alpha((uvw)^{-1})\alpha(b)\alpha(c)\alpha(a) = 0$

It is important here to discuss the *-ring of Laurent polynomials in x, with coefficients in a *-ring R, consists of all formal sums

$$f(x) = \sum_{i=k}^{m} a_i x^i,$$

with obvious addition and multiplication, where $a_i \in R$ and k, m are (possibly negative) integers, and with involution * defined as:

$$f^{*}(x) = \sum_{i=k}^{m} a_i x^i.$$

It is denoted as usual by $R[x; x^{-1}]$. If α is *-endomorphism of a *-ring R, then, the map $\tilde{\alpha}: R[x] \rightarrow R[x]$ is defined by

$$\tilde{\alpha}\left(\sum_{i=k}^{m} a_i x^i\right) = \sum\nolimits_{i=k}^{m} \alpha(a_i)\, x^i$$

for all i, is *-endomorphism of the polynomial *-ring $R[x]$, and it is clear this map extends α.

Corollary 15. *Let* R *be a *-ring, then,* $R[x]$ *is* α*-quasi-*-IFP* if and only if $R[x; x^{-1}]$ *is* $\tilde{\alpha}$*-quasi-*-IFP.*

Proof. It is sufficient to show the necessity. Clearly $S = \{1, x, x^2, \dots\}$ is a multiplicatively closed subset of $R[x]$. Since $R[x; x^{-1}] = S^{-1}R[x]$, it follows that $R[x; x^{-1}]$ is $\tilde{\alpha}$-quasi-*-IFP by **Proposition 16**.

Corollary 16. Let's suppose *R is a *-ring,* then *R[x] is* α *-*-reflexive* if and only if $R[x; x^{-1}]$ *is* $\tilde{\alpha}$*-*-reflexive.*

Proof. The important thing is to prove the necessity, since $R[x]$ is a *-subring of $R[x; x^{-1}]$. Clearly, $S = \{1, x, x^2, \dots\}$ is a multiplicatively closed subset of $R[x]$. Since $R[x; x^{-1}] = S^{-1}R[x]$, it follows that $R[x; x^{-1}]$ is $\tilde{\alpha}$-*-reflexive by **Proposition 17**.

A *-ring R is *-Armendariz (resp., quasi-*-Armendariz), when the polynomials

$$f(x) = \sum_{i=0}^{m} a_i x^i$$

and

$$g(x) = \sum_{j=0}^{n} b_j x^j \in R[x],$$

Satisfy

$$f(x)g(x) = f(x)g^{*}(x) = 0$$

(resp., $f(x)R[x]g(x) = f(x)R[x]g^{*}(x) = 0$), then, $a_i b_j = 0$ (resp., $a_i R b_j = 0$) for all i, j (consequently $a_i b_j^{*} = 0$ (resp.; $a_i R b_j^{*} = 0$)).

Theorem 1. *Let R be a *-Armendariz *-ring. Then the following statements are equivalent*

1. *R is α-quasi-*-IFP.*
2. $R[x]$ *is* $\tilde{\alpha}$*-quasi-*-IFP.*
3. $R[x; x^{-1}]$ *is* $\tilde{\alpha}$*-quasi-*-IFP.*

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Proof. It is enough to prove (1) to obtain (2). Let's consider

$$f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$$

Therefore,

$$f(x)R[x]g(x) = 0 = f(x)R[x]g^*(x).$$

By hypothesis,

$$a_i b_j = 0 = a_i b_j^*$$

and

$\alpha(a_i)\alpha(r)\alpha(b_j) \in R$ for all i, j and $r \in R$.

Hence, $\alpha(f(x))R[x]\alpha(g(x)) = 0$, and hence that $R[x]$ is $\tilde{\alpha}$-quasi-*-IFP.

Theorem 2. *Let us assume R is a quasi-*-Armendariz *-ring, then the following statements are equivalent.*

1. *R is* α*-*-reflexive.*
2. $R[x]$ *is* $\tilde{\alpha}$*-*-reflexive.*
3. $R[x; x^{-1}]$ *is* $\tilde{\alpha}$*-*-reflexive.*

Proof. It suffices to show that (1) $\Rightarrow$ (2). Assume

$$f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$$

leads us to obtain:

$$f(x)R[x]g(x) = 0 = f(x)R[x]g^*(x).$$

Since R is quasi-*-Armendariz, it is given $a_i R b_j = 0$ for all i, j. But, R is α-*-reflexive, so $\alpha(b_j)R\alpha(a_i) = 0$ for all i, j.

Consequently $\alpha(g(x))R[x]\alpha(f(x)) = 0$, and hence $R[x]$ is $\tilde{\alpha}$-*-reflexive.

The following corollaries are obtained by Theorems 1 and 2.

Corollary 17. *Let R be an Armendariz *-ring. Then, the following relations are equivalent:*

1. *R is* α*-quasi-*-IFP.*
2. $R[x]$ *is* $\tilde{\alpha}$*-quasi-*-IFP.*
3. $R[x; x^{-1}]$ *is* $\tilde{\alpha}$*-quasi-*-IFP.*

Corollary 18. *Let R be a quasi-Armendariz *-ring. Then, the following relations are equivalent:*

1. *R is* α*-*- reflexive.*
2. $R[x]$ *is* $\tilde{\alpha}$*-*-reflexive.*
3. $R[x; x^{-1}]$ *is* $\tilde{\alpha}$*-*-reflexive.*

It is known that, *the Dorroh extension* $D(R, Z) = \{(r, n): r \in R, n \in \mathbb{Z}\}$ of a *-ring R is a ring with componentwise addition and multiplication:

$$(r_1, n_1)(r_2, n_2) = (r_1 r_2 + n_1 r_2 + n_2 r_1, n_1 n_2).$$

The involution of R can be extended naturally to $D(R, \mathbb{Z})$ as $(r, n)^* = (r^*, n)$ (see (Aburawash, 1997)). A *-endomorphism α on R can be extended to $\tilde{\alpha}$ on $D(R, \mathbb{Z})$ by $\tilde{\alpha}(r, n) = (\alpha(r), n)$ (see (Başer et al., 2009)).

Proposition 18. A *-ring R is α-quasi-*-IFP with $\alpha(1) = 1$, if and only if its Dorroh extension $D(R, \mathbb{Z})$ is $\tilde{\alpha}$-quasi-*-IFP.

Proof. Let $(r_1, n_1), (r_2, n_2) \in D$ with $(r_1, n_1)(r_2, n_2) = 0 = (r_1, n_1)(r_2^*, n_2)$.

Then,

$r_1 r_2 + n_1 r_2 + n_2 r_1 + n_1 n_2 = 0 = r_1 r_2^* + n_1 r_2^* + n_2 r_1 + n_1 n_2$.

Since $\mathbb{Z}$ is a *-domain, we have $n_1 = 0$ or $n_2 = 0$. If $n_1 = 0$ then $0 = r_1 r_2 + n_2 r_1 = r_1(r_2 + n_2)$ and $0 = r_1 r_2^* + n_2 r_1 = r_1(r_2^* + n_2)$. Since R is α-quasi-*-IFP with $\alpha(1) = 1$, $0 = \tilde{\alpha}(r_1)\tilde{\alpha}(r)\tilde{\alpha}((r_2 + n_2)) = \alpha(r_1)\alpha(r)\alpha(r_2) + \alpha(r_1)\alpha(r)n_2$ and $0 = \tilde{\alpha}(r_1)\tilde{\alpha}(r)\tilde{\alpha}((r_2^* + n_2)) = \alpha(r_1)\alpha(r)\alpha(r_2{}^*) + \alpha(r_1)\alpha(r)n_2$ for all $r \in R$. This yields

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$0 = \tilde{\alpha}((r_1, n_1))\tilde{\alpha}((r, n))\tilde{\alpha}((r_2, n_2)) =$ $(\alpha(r_1)\alpha(r) + n\alpha((r_1))\,\alpha(r_2) +$ $(\alpha(r_1)\alpha(r) + n\alpha(r_1))n_2$ for any $(r, n) \in D$, and hence $\tilde{\alpha}((r_1, n_1))D\tilde{\alpha}((r_2, n_2)) = 0$. Now, let $n_2 = 0$. Then, $(r_1 + n_1)r_2 = 0$, and so $0 = (\alpha(r_1) + n_1)R\alpha(r_2) = 0$. It is similar to obtain $\tilde{\alpha}\big((r_1, n_1)\big)D\tilde{\alpha}\big((r_2, n_2)\big) = 0$, and thus, the Dorroh extension $D(R, \mathbb{Z})$ is $\tilde{\alpha}$-quasi-*-IFP.

Proposition 19. *A *-ring R is α-*-reflexive with $\alpha(1) = 1$, if and only if its Dorroh extension $D(R, \mathbb{Z})$ is $\tilde{\alpha}$-*-reflexive.*

Proof. Suppose that $(r_1, n_1), (r_2, n_2) \in D$ with $(r_1, n_1)(r, n)(r_2, n_2) = 0 =$ $(r_1, n_1)(r, n)(r_2^*, n_2)$. For any $(r, n) \in D$. The claim here is

$$\tilde{\alpha}\big((r_2, n_2)\big)\tilde{\alpha}\big((r, n)\big)\tilde{\alpha}\big((r_1, n_1)\big) = 0.$$

In fact, we have

$$(r_1rr_2 + n_1rr_2 + nr_1r_2 + n_1nr_2 + n_2r_1r +$$

$$n_1n_2r + nn_2r_1, n_1nn_2) = 0 = (r_1rr_2^* + n_1rr_2^*$$

$+nr_1r_2^* + n_1nr_2^* + n_2r_1r + n_1n_2r +$ $nn_2r_1, n_1nn_2)$, $r_1rr_2 + n_1rr_2 + nr_1r_2 +$ $n_1nr_2 + n_2r_1r + n_1n_2r + nn_2r_1 = 0$

$$= r_1rr_2^* + n_1rr_2^* + nr_1r_2^* + n_1nr_2^* + n_2r_1r + n_1n_2r + nn_2r_1$$

and

$$n_1nn_2 = 0,$$

since $\mathbb{Z}$ is a *-domain $n_1 = 0$, $n = 0$ or $n_2 = 0$. If $n_1 = 0$, then, $r_1rr_2 + nr_1r_2 + n_2r_1r +$ $nn_2r_1 = 0$, and so we have:

$$0 = r_1rr_2 + nr_1r_2 + n_2r_1r + nn_2r_1 = r_1(r, n)$$

$$(r_2, n_2) = (\alpha(r_2), n_2)(\alpha(r), n)\alpha(r_1) = \alpha(r_2)$$

$\alpha(r)\alpha(r_1) + \alpha(r_2)n\alpha(r_1) + n_2\alpha(r)\alpha(r_1) +$ $n_2n\alpha(r_1)$, since R is $\tilde{\alpha}$-*-reflexive *-ring. This shows that

$$\tilde{\alpha}\big((r_2, n_2)\big)\tilde{\alpha}\big((r, n)\big)\tilde{\alpha}\big((r_1, n_1)\big) = \alpha(r_2)\alpha(r)\alpha(r_1) + \alpha(r_2)n\alpha(r_1) + n_2\alpha(r)\,\alpha(r_1) + n_2n\alpha(r_1) = \alpha(r_2)\alpha(r)\alpha(r_1) + n_2\alpha(r)\alpha(r_1) + n\alpha(r_2)\alpha(r_1) + n_2n\alpha(r_1) + n_1\alpha(r_2)\alpha(r) + n_1n_2\alpha(r) + n_1n\alpha(r) = 0, \ n_2nn_1 = 0.$$

If $n_2 = 0$, then,

$$r_1rr_2 + n_1rr_2 + nr_1r_2 + n_1nr_2 = (r_1, n_1)(r, n)r_2 = \alpha(r_2)$$

$$\tilde{\alpha}\big((r, n)\big)\tilde{\alpha}\big((r_1, n_1)\big) = \alpha(r_2)\alpha(r)\alpha(r_1) + \alpha(r_2)\alpha(r)n_1 + \alpha(r_2)n\alpha(r_1) + \alpha(r_2)nn_1 =$$

$$\alpha(r_2)\alpha(r)\alpha(r_1) + n_2\alpha(r)\alpha(r_1) + n\alpha(r_2)\alpha(r_1) + n_2n\alpha(r_1) + n_1\alpha(r_2)\alpha(r) + n_1n_2\alpha(r) + n_1n\alpha(r_2) = 0.$$

Therefore, we get $0 = \tilde{\alpha}\big((r_2, n_2)\big)\tilde{\alpha}\big((r, n)\big)$ $\tilde{\alpha}\big((r_1, n_1)\big)$, then, D is $\tilde{\alpha}$-*-reflexive.

Conversely, assume that D is $\tilde{\alpha}$-*-reflexive. Let $\alpha(r_1)R\alpha(r_2) = 0 = \alpha(r_1)R\alpha(r_2^*)$, for $r_1, r_2 \in R$. Then, $\alpha(r_1)(\alpha(r) + n))\alpha(r_2) = 0 =$ $\alpha(r_1)(\alpha(r) + n))\alpha(r_2^*)$ for any $(r, n) \in D$, this implies:

$0 = \tilde{\alpha}\big((r_1, 0)\big)\tilde{\alpha}\big((r, n)\big)\tilde{\alpha}\big((r_2, 0)\big) =$ $\tilde{\alpha}\big((r_1, 0)\big)\tilde{\alpha}\big((r, n)\big)\tilde{\alpha}\big((r_2^*, 0)\big)$, for any $(r, n) \in D$. Since D is $\tilde{\alpha}$-*-reflexive, we have $0 =$ $\tilde{\alpha}\big((r_2, 0)\big)\tilde{\alpha}\big((r, n)\big)\tilde{\alpha}\big((r_1, 0)\big) =$ $\tilde{\alpha}\big((r_2^*, 0)\big)\tilde{\alpha}\big((r, n)\big)\tilde{\alpha}\big((r_1, 0)\big)$,

and hence, $\alpha(r_2)(\alpha(r) + n)\alpha(r_1) = 0 =$

$\alpha(r_2^*)(\alpha(r) + n)\alpha(r_1)$, thus,

$$\alpha(r_2)R\alpha(r_1) = 0 = \alpha(r_2^*)R\alpha(r_1).$$

Recall that, a ring R is called right Ore, if it is given $a, b \in R$ with b regular, there exist $a_1, b_1 \in R$ with b_1 regular such that: $ab_1 =$ ba_1. Left Ore is defined similarly, and R is Ore ring, whether it is both right and left Ore. For

ISSN: online 2617-2186 print 2617-2178

*-rings, right Ore implies left Ore and vice versa. It is a known fact that R is Ore if and only if its classical quotient ring Q of R exists and for *-rings, * can be extended to Q by $(a^{-1}b)^* = b^*(a^*)^{-1}$. A *-automorphism α on Q by $\tilde{\alpha}(ab^{-1}) = \alpha(a)\alpha(b)^{-1}$ (see [(Martindale & 3rd, 1969), Lemma 4], (Hong et al., 2006)). The following theorem is generalized by [(Aburawash & Abdulhafed, 2018a), Theorem 5].

Theorem 3. *Let* R *be an Ore *-ring,* α *the *-automorphism of* R*, and* Q *be its classical quotient *-ring, then,* R *is* α*-*-reflexive if and only if* Q *is* $\tilde{\alpha}$*-*-reflexive.*

Proof. Let R be an α-*-reflexive *-ring and $\beta\gamma\xi = 0 = \beta\gamma\xi^*$ with $\beta = au^{-1}$, $\gamma = bv^{-1}$ and $\xi = cw^{-1} \in Q$. By hypothesis for $a, b, c, d, u, v, w \in R$, there exist $b_1, u_1 \in R$ with u_1 regular

$$ub_1 = bu_1, u^{-1}b = b_1u_1^{-1}, \qquad (1)$$

then, $0 = \beta\gamma\xi = au^{-1}bv^{-1}cw^{-1} = ab_1u_1^{-1}v^{-1}cw^{-1}$. Also, for $c, v \in R$ there exist $c_1, v_1, c_1^* \in R$ with v_1 regular such that:

$$vc_1 = cv_1, v^{-1}c = c_1v_1^{-1}$$

and

$$vc_1^* = c^*v_1, v^{-1}c^* = c_1^*v_1^{-1}, \qquad (2)$$

so, we have $0 = \beta\gamma\xi = ab_1u_1^{-1}c_1v_1^{-1}w_1^{-1}$. For $c, u \in R$, there exist $c_1^*, c_2, c_2^*, u_2 \in R$ with u_2 regular such that:

$$u_1c_2 = c_1u_2, u_1^{-1}c_1 = c_2u_2^{-1}, \qquad (3)$$

$$u_1c_2^* = c_1^*u_2, u_1^{-1}c_1^* = c_2^*u_2^{-1}, \qquad (4)$$

and hence, it is obtained

$$0 = \beta\gamma\xi = ab_1c_2u_2^{-1}v_1^{-1}w^{-1} = ab_1c_2(wv_1u_2)^{-1}$$

which implies $ab_1c_2 = 0$. Similarly,

$$0 = \gamma\beta\xi^* = au^{-1}bv^{-1}(cw^{-1})^* = au^{-1}bv^{-1}(w^*)^{-1}c^* = ab_1u_1^{-1}v^{-1}(w^*)^{-1}c^* = ab_1(w^*vu_1)^{-1}c^*,$$

where $g = w^*vu_1 \in R$. For $c, g \in R$, there exist $g_1, c_2^* \in R$ with g_1 regular such that:

$$gc_2^* = c^*g_1,\ g^{-1}c^* = c_2^*g_1^{-1}.$$

Thus,

$$0 = \gamma\beta\xi^* = ab_1c_2^*g_1^{-1},$$

for which implies that $ab_1c_2^* = 0$, then, $ab_1c_2 = 0 = ab_1c_2^*$. In the following computations, the condition that R is α-*-reflexive can be used freely. Since $ab_1c_2 = 0$, we have $c_2b_1a = 0$, and so $c_2b_1au = 0$. This implies that:

$aub_1c_2 = abu_1c_2 = 0$ by **(1),** then, $u_1c_2ba = 0$, and so $c_1u_2ba = 0$ by **(3)**. This leads to $abc_1u_2 = 0$, and $abc_1 = 0$. Hence, $c_1ba = 0$, and thus $vc_1ba = 0$, so $abcv_1 = 0$. Hence, we get $abc = 0$, and so $cba = 0$. Similarly, $ab_1c_2^* = 0$, we have $c_2^*b_1a = 0$, and so $c_2^*b_1au = 0$. This implies that $aub_1c_2^* = abu_1c_2^* = 0$ by **(1),** then, $u_1c_2^*ba = 0$, and so $c_1^*u_2ba = 0$ **by (4)**. This shows that $abc_1^*u_2 = 0$ and $abc_1^* = 0$. Hence, $c_1^*ba = 0$, and thus $vc_1^*ba = 0$, so $abc^*v_1 = 0$ **by (2)**. Hence, we get $abc^* = 0$ and so $c^*ab=0$. On the other hand, $\xi\gamma\beta = \tilde{\alpha}(cw^{-1})\tilde{\alpha}(bv^{-1})\tilde{\alpha}(au^{-1})$ and similarly there exist $a_3, a_4, b_3, w_3, v_3, v_4 \in R$ with w_3, v_3, v_4 regular such that:

$$\alpha(w)\alpha(b_3) = \alpha(b)\alpha(w_3), \qquad \alpha(w^{-1})\alpha(b) = \alpha(b_3)\alpha(w_3^{-1}),$$

$$\alpha(v)\alpha(a_3) = \alpha(a)\alpha(v_3), \qquad \alpha(v^{-1})\alpha(a) = \alpha(a_3)\alpha(v_3^{-1})$$

and

$$\alpha(w)\alpha(a_4) = \alpha(a_3)\alpha(v_4), \quad \alpha(w_3^{-1})\alpha(a_3) = \alpha(a_4)\alpha(v_4^{-1}).$$

Then, we get

$$\xi\gamma\beta = \alpha(c)\alpha(b_3)\alpha(w_3^{-1})\alpha(a_3)\alpha(v_3^{-1})\alpha(u^{-1}) = \alpha(c)\alpha(b_3)\,\alpha(a_4)\alpha(v_4^{-1})\alpha(v_3^{-1})\alpha(u^{-1}) =$$

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$\alpha(cb_3a_4)\alpha\left((uv_3v_4)^{-1}\right)$. Since $\alpha(cba) = 0$, we obtain $\alpha(w_3)\alpha(cba) = 0$, and so

$\alpha(a)\alpha(b)\alpha(w_3)\alpha(c) = 0 =$
$\alpha(a)\alpha(w)\alpha(b_3)\alpha(c)$.

This implies that:

$$\alpha(c)\alpha(b_3)\alpha(a)\alpha(w) = 0,$$

and thus

$\alpha(c)\alpha(b_3)\alpha(a) = 0$.
Then,

$\alpha(c)\alpha(b_3)\alpha(a)\alpha(v_3) =$
$\alpha(c)\alpha(b_3)\alpha(v)\alpha(a_3) = 0$ and so,
$\alpha(v)\alpha(a_3)\alpha(b_3)\alpha(c) = 0$.

Hence, $\alpha(a_3b_3c) = 0 = \alpha(cb_3a_3)$ and then

$\alpha(c)\alpha(b_3)\alpha(a_3)\alpha(v_4) =$
$\alpha(c)\alpha(b_3)\alpha(w)\,\alpha(a_4) =$
$\alpha(wa_4)\alpha(b_3)\alpha(c) = 0$.

It follows that $\alpha(a_4)\alpha(b_3)\alpha(c) = 0$ and we get $\alpha(c)\alpha(b_4)\alpha(a_3) = 0$. Therefore,

$\xi\gamma\beta = \tilde{\alpha}(cw^{-1}bv^{-1}au^{-1}) =$
$\alpha(cb_3a_4)(uv_3v_4)^{-1} = 0$.Then Q $\tilde{\alpha}$-*-reflexive.

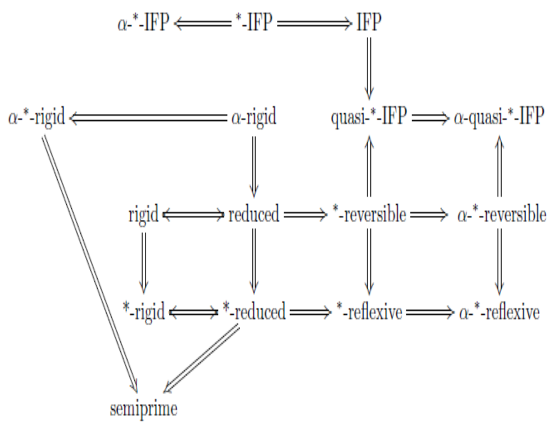
Finally, according to the study (Abdulhafed, 2019)], [(Aburawash & Abdulhafed, 2018a), Theorem 5], and **Theorem 3**, the following corollaries are deduced.

Corollary 19. *If* R is reflexive **-ring,* then Q *is $\tilde{\alpha}$-*-reflexive.*
Corollary 20. *If* Q is reflexive **-ring, then* R *is* α-**-reflexive.*
Corollary 21. *If* R *is* α-**-reversible *-ring, then* Q *is $\tilde{\alpha}$-*-reflexive.*

Corollary 22. If Q is $\tilde{\alpha}$-*-reversible *-ring, then R is α-*-reflexive.

Conclusion And Future Directions

All the results of the paper are summarized by using a diagram explaining the relations among the corresponding classes.It is possible to follow implications in the class of rings with involution.



Most importantly, the study has explored the following several new results.

1. Every α-rigid is α-**-reflexive.*
2. Every *-rigid is α-**-reflexive.*
3. Every α-*-reversible is α-**-reflexive.*
4. Each α-*-reversible is α-quasi-*-IFP.

Finally, these results pave the way for further research into some classes of *-rings which generalize that of α-*-reflexive *-rings and investigate their properties.

Based on the above results, we strongly recommend studying the *-rings independently by using both morphisms and structures presiding its involution. However, it is good to join the involutive structures with their non-involutive counterparts. Future work will be concerned with deeper analysis of *-rings, new properties of the created structures and new several structures in the involutive sense. The following questions should be an inception for new works:
1. Studying the extensions over *-Baer *-rings as polynomials, skew polynomials, and matrices is very important. What are the hypotheses needed to do that?
2. How are the extensions of α -*-reflexive possible?
3. Is there a possibility to get a definition for

ISSN: online 2617-2186 print 2617-2178

central α-*-reflexive *-rings analogously to central *-reflexive rings?

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ISSN: online 2617-2186 print 2617-2178

مجلة المختار للعلوم 36 (1): 42-56، 2021

تمديد الحلقات الالتفافية الانعكاسية

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تاريخ الاستلام: 13 يناير 2021 / تاريخ القبول: 31 يناير 2021
Doi: https://doi.org/10.54172/mjsc.v36i1.22

المستخلص: تقدم الدراسة مفهوم تمديد (خاصية شبة إدراج المعاملات، الانعكاس) للحلقات الالتفافية كونه تعميما على (خاصية شبه إدراج المعاملات، الانعكاس) للحلقات الالتفافية وكل (خاصية شبه إدراج المعاملات، الانعكاس) للحلقات الالتفافية تكون تمديدا (لخاصية شبه إدراج المعاملات، الانعكاس) للحلقات الالتفافية. كما تدرس الشرط الكافي لخاصية شبه إدراج المعاملات والانعكاس لتكون تمديدا (لخاصية شبه إدراج المعاملات، الانعكاس). ويتم أخيرا التحقق من تمديد (لخاصية شبه إدراج المعاملات، الانعكاس) لبعض أنواع الحلقات كثيرات الحدود

الكلمات المفتاحية: الحلقات الالتفافية المختزلة، الحلقات الالتفافية الصلبة، تمديد الحلقات الالتفافية الصلبة، تمديد الحلقات الالتفافية بخاصية إدراج المعاملات، تمديد الحلقات الالتفافية بخاصية شبة إدراج المعاملات، تمديد الحلقات الالتفافية الانقلابية، تمديد الحلقات الالتفافية الانعكاسية.

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