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On the Existence of A Unique Solution for Nonlinear Ordinary Differential Equations of Order m

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Abstract

In this work I state and prove a theorem for local existence of a unique solution for the Nonlinear Ordinary Differential Equations (NODE):

$$x^{(m)}(t) = f(t, x(t), x'(t), x''(t), \ldots, x^{(m-1)}(t)) \tag{1}$$

of order m; where m is a positive integer; having the initial conditions:

$$x^{(j)}(a) = c_j, \; j = 0, 1, \ldots, m-1 \quad , \quad x^{(0)}(a) = x(a) = c_0 \tag{2}$$

Since the (NODE) (1) with the initial conditions (2) is equivalent to the Integral Equation:

$$x(t) = c_0 + \sum_{j=1}^{m-1} c_j \frac{(t - t_0)^j}{j!} + \int_a^t \int_a^{s_1} \int_a^{s_2} \cdots \int_a^{s_{m-2}} \int_a^{s_{m-1}} f(s_m, x(s_m), x'(s_m), x''(s_m), \ldots, x^{(m-1)}(s_m)) ds_m ds_{m-1} ds_{m-2} \ldots ds_2 ds_1 \tag{3}$$

We denote the right hand side (r.h.s.) of (3) by the nonlinear operator $Q(x)t$; then prove that this operator is contractive in a metric space E subset of the Banach space B of the class of continuous bounded functions $x(t) \in C^m(\mathbb{R})$ defined by:

$$\mathbf{B} = \left\{ (t, x(t), x'(t), \ldots, x^{(m-1)}(t)) \mid |t-a| < \infty \, , \, \left| x^{(j)}(t) - c_j \right| < \infty \right\} \tag{4}$$

and B is equipped with the weighted norm:

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$$\|x\| = \max_{|t-a|\le T_m} \left(e^{-\nu L|t-a|} \sum_{j=0}^{m-1} \left| x^{(j)}(t) \right| \right) \tag{5}$$

which is known as Bielescki's type norm. $\nu \ge 2$, $L = \max(l,1)$ are finite real numbers, where $l > 0$ is the Lipschitz coefficient of the r.h.s. of (1) in B1(a subset of the Banach space B given by (4)) defined by:

$$\mathbf{B1} = \left\{ (t, x(t), x'(t), \ldots, x^{(m-1)}(t)) \mid |t-a| \le T_m \ , \ \left| x^{(j)}(t) - c_j \right| \le T_j \right\} \tag{6}$$

Where T_j for $j = 0, 1, \ldots, m-1$,and T_m are finite real numbers.

Key Words: Nonlinear Ordinary Differential Equation of Order m; Banach Space of Bounded Functions $x(t) \in C^m(\mathbb{R})$; Lipschitz Condition; Contraction Mapping Theorem; Existence of a Unique Solution Globally.

Introduction

When the function f in the r.h.s. of (1) depends linearly on its arguments except t then equation (1) is an m$^{\underline{th}}$ order linear ordinary differential equation and to prove the Existence of a Unique solution for it in $[a-T_m, a+T_m]$ one usually write down its equivalent system consisting of m equations of first order and use one of the well known theorems to prove the existence of a unique solution for $t \in [a, a+\delta]$ then mimic the same steps of the proof for $t \in [a-\delta, a]$; after that use another theorem to show whether the solution do exist for all $t \in [a-T_m, a+T_m]$ or not as in (Hurewicz, 1974), and when the m$^{\underline{th}}$ order differential equation is nonlinear one may face difficulties in dealing with its equivalent system of first order equations. But by the theorem which I am going to state and prove in this paper one can easily prove the existence of a unique solution for an m$^{\underline{th}}$ order nonlinear ordinary differential equation on the general form (1) for all $t \in [a-T_m, a+T_m]$ directly in the very simple metric space consisting of the functions $x(t) \in C^m[a-T_m, a+T_m]$ and subset of the Banach space (4) (Hutson and Pym, 1980) equipped with the simple efficient norm (5) (Bojeldain, 1995), which is a simple modification on the Bielescki's type norm $\sup_t (e^{-r(t)} x(t))$ used in (Bieleski, 1956). Moreover if the Lipschitz condition (7) is guaranteed to be satisfied in the Banach space (4), then the theorem guarantees the existence of a unique solution for $|t-a| < \infty$ <u>in most</u> cases and not in general as mentioned in (Jankó, 1990) for the case of the single first order nonlinear ODE $x'(t) = f(t, x(t))$.

Note that this theorem is valid for m[th] order linear ordinary differential equations as well.

Theorem

Let us have the (NODE) (1) with the initial conditions (2) and suppose that the function f in the r.h.s. of (1) is continuous and satisfies the Lipschitz condition:

$$\left| f(t, x(t), x'(t), x''(t), ..., x^{(m-1)}(t)) - f(t, y(t), y'(t), y''(t), ..., y^{(m-1)}(t)) \right| \leq l \sum_{j=0}^{m-1} \left| x^{(j)}(t) - y^{(j)}(t) \right| \qquad (7)$$

in B1 given by (6); then the initial value problem (1) and (2) has a unique solution in the (m+1)-dimensional metric space E(of the functions $x(t) \in C^m[a-\delta, a+\delta]) \subseteq \mathbf{B}$ defined by:

$$\mathbf{E} = \left\{ (t, x(t), x'(t), ..., x^{(m-1)}(t)) \mid \left| t-a \right| \leq \delta \ , \ \left| x^{(j)}(t) - c_j \right| \leq T \right\} \qquad (8)$$

Such that $\delta = \min\left(a \ , \frac{T}{M}\right)$; where $T = \min(T_j, T_m)$ for $j = 0, 1, ..., m-1$,

$M = M_2 \sum_{j=1}^{m} \frac{|t-a|^{j-1}}{j!}$, $M_2 = \max\left(\left| c_j \right| \ , \ M_1 \right)$, M_1 is the upper bound of $|f|$ in B1 i.e.:

$$\left| f(t, x(t), x'(t), ..., x^{(m-1)}(t)) \right| \leq M_1 \quad \forall (t, x(t), x'(t), ..., x^{(m-1)}(t)) \in \mathbf{B1}$$
(9)

Proof

Integrating both sides of (1) from a to t m-times and using the initial conditions (2) we obtain the integral equation:

$$x(t) = c_0 + \sum_{j=1}^{m-1} c_j \frac{(t-a)^j}{j!} + \int_a^t \int_a^{s_1} \int_a^{s_2} \cdots \int_a^{s_{m-2}} \int_a^{s_{m-1}} f(s, x(s), x'(s), x''(s), ..., x^{(m-1)}(s)) ds \, ds_{m-1} ... ds_2 ds_1 \qquad (10)$$

To form a fixed point problem $x(t) = Q(x)t$ denote the r.h.s. of (10) by $Q(x)t$, and to apply the contraction mapping theorem we first show that $Q : \mathbf{E} \rightarrow \mathbf{E}$; then prove that Q is contractive in E.

We see that:

$$| Q(x)t - c_0 | \leq \sum_{j=1}^{m-1} \left| c_j \frac{(t-a)^j}{j!} \right| + \left| \int_a^t \int_a^{s_1} \int_a^{s_2} \ldots \int_a^{s_{m-2}} \int_a^{s_{m-1}} \right| f(s, x(s), x'(s), x''(s), \ldots, x^{(m-1)}(s)) \Big| ds ds_{m-1} ds_{m-2} \ldots ds_2 ds_1 | \leq$$

$$\leq \sum_{j=1}^{m-1} | c_j | \frac{|t-a|^j}{j!} + M_1 \frac{|t-a|^m}{m!} \leq M_2 \sum_{j=1}^{m} \frac{|t-a|^j}{j!} \tag{11}$$

Therefore:

$$| Q(x)t - c_0 | \leq M_2 \sum_{j=1}^{m} \frac{|t-a|^j}{j!} = M_2 |t-a| \sum_{j=1}^{m} \frac{|t-a|^{j-1}}{j!} \leq M\delta \leq T \tag{12}$$

which means that $Q : \mathbf{E} \to \mathbf{E}$.

To prove that Q is contractive we consider the difference:

$$| Q(x)t - Q(y)t | = | Q(x) - Q(y) | (t) \leq$$
$$\left| \int_a^t \int_a^{s_1} \int_a^{s_2} \ldots \int_a^{s_{m-2}} \int_a^{s_{m-1}} \Big| f(s, x(s), x'(s), \ldots, x^{(m-1)}(s)) - \right.$$
$$\left. - f(s, y(s), y'(s), \ldots, y^{(m-1)}(s)) \Big| ds\, ds_{m-1} ds_{m-2} \ldots ds_2 ds_1 \right| \tag{13}$$

which according to Lipschitz condition (7) yields:

$$| Q(x) - Q(y) | (t) \leq$$
$$\leq \left| \int_a^t \int_a^{s_1} \int_a^{s_2} \ldots \int_a^{s_{m-2}} \int_a^{s_{m-1}} l \sum_{j=0}^{m-1} \left| x^{(j)}(s) - y^{(j)}(s) \right| ds\, ds_{m-1} ds_{m-2} \ldots ds_2 ds_1 \right| \tag{14}$$

Multiply the r.h.s. of (14) by $e^{-\nu L|t-a|}$ $e^{\nu L|t-a|}$ and get:

$$| Q(x) - Q(y) | (t) \leq$$
$$\leq \left| \int_a^t \int_a^{s_1} \int_a^{s_2} \ldots \int_a^{s_{m-2}} \int_a^{s_{m-1}} l \sum_{j=0}^{m-1} \left| x^{(j)}(s) - y^{(j)}(s) \right| e^{-\nu L|s-a|} \times \right.$$
$$\left. \times e^{\nu L|s-a|} ds\, ds_{m-1} ds_{m-2} \ldots ds_2 ds_1 \right| \tag{15}$$

Inequality (15) leads to:

$$\left| Q(x)-Q(y) \right|(t) \le$$

$$L\left|\int_a^t\int_a^{s_1}\int_a^{s_2}\cdots\int_a^{s_{m-2}}\int_a^{s_{m-1}}\left(\max_{|s-a|\le\delta}\left(e^{-\nu L|s-a|}\sum_{j=0}^{m-1}\left|x^{(j)}(s)-y^{(j)}(s)\right|\right)\times\right.\right.$$

$$\left.\left.\times\ e^{\nu L|s-a|}\right)\,ds\,ds_{m-1}ds_{m-2}\ldots ds_2ds_1\right| \tag{16}$$

According to (5), the norm definition, inequality (16) becomes:

$$\left| Q(x)-Q(y) \right|(t) \le$$

$$\le L\|x-y\|\left|\int_a^t\int_a^{s_1}\int_a^{s_2}\cdots\int_a^{s_{m-2}}\int_a^{s_{m-1}}e^{\nu L|s-a|}\,ds\,ds_{m-1}ds_{m-2}\ldots ds_2ds_1\right| \tag{17}$$

Manipulating the integrals in (17) we obtain the following inequality:

$$\left| Q(x)-Q(y) \right|(t) \le$$

$$\le L\|x-y\|\left|\frac{1}{(\nu L)^m}\left(e^{\nu L|t-a|}-1\right)-\sum_{j=1}^{m-1}\frac{|t-a|^j}{j!(\nu L)^{m-j}}\right| \le$$

$$\le L\|x-y\|\frac{1}{(\nu L)^m}\left(e^{\nu L|t-a|}-1\right)+\sum_{j=1}^{m-1}\frac{(\nu L|t-a|)^j}{j!(\nu L)^m} \le$$

$$\le L\|x-y\|\frac{1}{(\nu L)^m}\left(e^{\nu L|t-a|}-1\right)+\sum_{j=1}^{\infty}\frac{(\nu L|t-a|)^j}{j!(\nu L)^m} =$$

$$= L\|x-y\|\frac{2}{(\nu L)^m}\left(e^{\nu L|t-a|}-1\right) \tag{18}$$

i.e.

$$\left| Q(x)-Q(y) \right|(t) \le L\|x-y\|\frac{2}{(\nu L)^m}\left(e^{\nu L|t-a|}-1\right) \tag{19}$$

Multiplying both sides of (19) by $e^{-\nu L|t-a|}$ leads to:

$$e^{-\nu L|t-a|}\left| Q(x)-Q(y) \right|(t) \le \frac{2}{\nu(\nu L)^{m-1}}\left(1-e^{-\nu L|t-a|}\right)\|x-y\| \le$$

$$\le \frac{2}{\nu(\nu L)^{m-1}}\left(1-e^{-\nu L\delta|}\right)\|x-y\| \tag{20}$$

The most r.h.s. of (20) is independent of t, thus it is an upper bound for its l.h.s. for any $|t-a|\le\delta$; whence:

$$\max_{|t-a|\le\delta}\left(e^{-\nu L|t-a|}\left|Q(x)-Q(y)\right|(t)\right)\le\frac{2}{\nu(\nu L)^{m-1}}\left(1-e^{-\nu L\delta|}\right)\|x-y\| \tag{21}$$

which, according to the norm definition (5), gives:

$$\|Q(x)-Q(y)\|\le\frac{2}{\nu(\nu L)^{m-1}}\left(1-e^{-\nu L\delta|}\right)\|x-y\|\le$$

$$\left(1-e^{-\nu L\delta|}\right)\|x-y\| \tag{22}$$

noting that for finite $L\ge1$, $\nu\ge2$, and $m\ge1$ we have

$$\frac{2}{\nu(\nu L)^{m-1}}\le\frac{2}{\nu}\le1.$$

Since $0<\left(1-e^{-\nu L\delta)}\right)<1$; then $Q(x)t$ is a contraction operator in E and thus has a unique solution for $t\in[a-\delta,a+\delta]$.

Conclusion

We see that the contraction coefficient $0<\left(1-e^{-\nu L\delta)}\right)<1$ for any finite $\delta>0$ which means that the solution for the problem under consideration is, in fact, guaranteed globally for $|t-a|\le T_m$ and not only locally for $|t-a|\le\delta$. Moreover; in most cases; if the function f in the r.h.s. of (1) is continuous and satisfies Lipschitz condition in the Banach space (5) with finite positive Lipschitz coefficient then the theorem is proved for t in any interval I of finite length because the contraction coefficient will be positive and less than $\left(1-e^{-\nu L\mu(I)}\right)<1$; where $\mu(I)$ is the measure of the interval I.

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On The Existence of A Unique Solution For Nonlinear Ordinary Differential Equations of Order m

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الملخص

في هذا البحث أقدم نظرية تضمن وجود حل وحيد موضعيا للمعادلات التفاضلية العادية غير الخطية :

$$x^{(m)}(t) = f(t, x(t), x'(t), x''(t), \ldots, x^{(m-1)}(t)) \quad (1)$$

ذات رتبة m ، حيث m عدد صحيح موجب ، مع الشروط الابتدائية :

$$x^{(j)}(a) = c_j, \; j = 0, 1, \ldots, m-1 \quad , \quad x^{(0)}(a) = x(a) = c_0 \quad (2)$$

طالما أن المعادلة (1) مع الشروط الإبتدائية (2) تكون مكافئة للمعادلة التكاملية :

$$+\int_a^t \int_a^{s_1} \int_a^{s_2} \cdots \int_a^{s_{m-2}} \int_a^{s_{m-1}} f(s_m, x(s_m), x'(s_m), x''(s_m), \ldots$$
$$\ldots, x^{(m-1)}(s_m)) ds_m ds_{m-1} ds_{m-2} \ldots ds_2 ds_1 \quad (3)$$

نرمز للطرف الأيمن للمعادلة (3) بالمؤثر غير الخطي $Q(x)t$ ثم نثبت أن هذا المؤثر يكون تقليصي (contractive) في فضاء متري E جزئي من فضاء بناخ B المكون من فصيلة الدوال المتصلة المحدودة $x(t) \in C^m$(¡) والمعرف كما يلي :

$$\mathbf{B} = \left\{ (t, x(t), x'(t), \ldots, x^{(m-1)}(t)) \mid |t-a| < \infty \, , \, \left| x^{(j)}(t) - c_j \right| < \infty \right\} \quad (4)$$

المزود بالمعيار الموزون :

$$\|x\| = \max_{|t-a| \le T_m} \left(e^{-\nu L |t-a|} \sum_{j=0}^{m-1} \left| x^{(j)}(t) \right| \right) \quad (5)$$

والذي يعرف باسم معيار بيالسكي حيث $\nu \geq 2$ ، $L = \max(l,1)$ أعداد حقيقية منتهية ، $l > 0$ هو ثابت ليبشتز للدالة $f(t, x(t), x'(t), x''(t), ..., x^{(m-1)}(t))$ في مجموعة B1 جزئية من فضاء بناخ B و معرفة كما يلي:

$$\mathbf{B1} = \left\{ (t, x(t), x'(t), ..., x^{(m-1)}(t)) \mid |t-a| \leq T_m \ , \ \left| x^{(j)}(t) - c_j \right| \leq T_j \right\} \quad (6)$$

مع مراعاة أن T_m ، T_j أعداد حقيقية منتهية لجميع القيم $j = 0,1,...,m-1$ ، والفضاء المتري E معرف كما يلي:

$$\mathbf{E} = \left\{ (t, x(t), x'(t), ..., x^{(m-1)}(t)) \mid |t-a| \leq \delta \ , \ \left| x^{(j)}(t) - c_j \right| \leq T \right\} \quad (7)$$

حيث $\delta = \min\left(a \, , \frac{T}{M}\right)$ ، $T = \min(T_j, T_m)$ لقيم $j = 0,1,...,m-1$ ،

$M = M_2 \sum_{j=1}^{m} \frac{|t-a|^{j-1}}{j!}$ ، $M_2 = \max\left(\ |c_j| \ , \ M_1 \ \right)$ ، M_1 يحقق:

$$\left| f(t, x(t), x'(t), ..., x^{(m-1)}(t)) \right| \leq M_1 \quad \forall (t, x(t), x'(t), ..., x^{(m-1)}(t)) \in \mathbf{B1} \quad (8)$$