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Some Extensions and Generalizations of Kümmer's Third Summation Theorem

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Abstract: The motive of this research paper is to obtain explicit forms of certain extensions and generalizations of Kümmer's third summation theorem, which have not previously appeared in the literature, by using the summation theorem given by Rakha and Rathie (2011). The results derived in this paper are interesting and may be beneficial.

بعض الامتدادات والتعميمات لنظرية التجميع الثالثة لكومر

الكلمات المفتاحية : الدوال الفوق هندسية؛
نظرية التجميع الثالثة لكومر؛
نظريات التجميع الفوق هندسية.

المستخلص : دوافع هذا البحث هي الحصول على أشكال صريحة من بعض الامتدادات والتعميمات لنظرية (كومار) Kümmer الثالثة للتجميع، باستخدام نظرية التجميع التي قدمها سنة (2011) (رخا) Rakha و(راثي) Rathie والتي لم تظهر في مؤلفات سابقة. النتائج المستمدة من هذا البحث مثيرة للاهتمام وقد تكون مفيدة.

INTRODUCTION and PRELIMINARIES

The theory of hypergeometric functions is rapidly developing with a large number of applications in the real world. The major development was given (although Euler and Pfaff had found many important results) by Gauss and Kümmer on the series ${}_2F_1$ and ${}_1F_1$ respectively and the other higher order hypergeometric functions such as ${}_2F_1$ and ${}_3F_2$ presented by (Clausen, 1828; Goursat, 1883), respectively.

- A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha, \beta; \gamma; z]$ is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series:

$$ {}_pF_q\left[\begin{matrix}(\alpha_p);\\ \\ (\beta_p);\end{matrix}\quad z\right] = {}_pF_q\left[\begin{matrix}\alpha_1,\alpha_2,\ldots\alpha_p;\\ \\ \beta_1,\beta_2,\ldots\beta_p;\end{matrix}\quad z\right] $$

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$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_n \quad z^n}{\prod_{j=1}^{q} (\beta_j)_n \quad n!}, \qquad (1.1)$$

is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here p and q are positive integers or zero, and we assume that the variable z , the numerator parameters $\alpha_1, \alpha_2, ...\alpha_p$ and the denominator parameters $\beta_1, \beta_2, ...\beta_q$ take on complex values, provided that

$$\beta_j \neq 0, -1, -2, ... \; ; \; j =, 1, 2, ...q. \qquad (1.2)$$

Supposing that none of the numerator and denominator parameters is zero or a negative integer, we note that the ${}_pF_q$ series defined by equation (1.1):

(i) Convergence for $|z| < \infty$, if $p \le q$;

(ii) Convergence for $|z| < 1$, if $p \le q + 1$;

(iii) Divergence for all $z, z \neq 1$, if $p > q + 1$;

(iv) Convergence absolutely for $|z| = 1$, if $p = q + 1$ and $\Re(w) > 0$;

(v) Converges conditionally for $|z| = 1$ $(z \neq 1)$ if $p = q + 1$ and $-1 < \Re(w) \le 0$;

(vi) Divergence for $|z| = 1$, if $p = q + 1$ and $\Re(w) \le -1$.

Where, by convention, a product over an empty set is interpreted as 1 and

$$w := \sum_{j=1}^{q} \beta_i - \sum_{j=1}^{p} \alpha_i . \qquad (1.3)$$

In this paper, we shall use the following standard notations:

$\mathbb{N} := \{1, 2, 3, ...\}$; $\mathbb{N} := \mathbb{N}_0 \cup \{0\}$;

$\mathbf{Z}_0^- ; = \mathbf{Z}^- \cup \{0\} = \{0, -1, -2, ...\}$. The symbols $\mathbb{C}, \mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers, respectively.

- The Pochhammer symbol $(\alpha)_p$ [(Rainville, 1971), p.(22), Eq.(1), Q. N.(8) and Q. N.(9), see also (Srivastava & Manocha, 1984), p.(23), Eq.(22) and Eq.(23)] is defined by:

$$(\alpha)_p := \frac{\Gamma(\alpha + p)}{\Gamma(\alpha)}$$

$$= \begin{cases} 1 & ;(p = 0; \alpha \in \mathbb{C} \setminus \{0\}), \\ \alpha(\alpha+1)...(\alpha+n-1) & ;(p = n \in \mathbb{N}; \alpha \in \mathbb{C}), \\ \dfrac{(-1)^k n!}{(n-k)!} & ;(\alpha = -n; p = k; n, k \in \mathbb{N}_0; 0 \le k \le n), \\ 0 & ;(\alpha = -n; p = k; n, k \in \mathbb{N}_0; k > n), \\ \dfrac{(-1)^k}{(1-\alpha)_k} & ;(p = -k; k \in \mathbb{N}; \alpha \in \mathbb{C} \setminus \mathbb{Z}), \end{cases}$$

it being understood conventionally that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists (see, for details, [(Srivastava & Manocha, 1984), p.21 *et seq.*]). Here, we aim at the extensions and generalizations of Kümmer's third summation theorem involving the summation theorems given by Rakha and Rathie. Here, for the purpose of the present investigation, we would like to recall the following summation formula which is due to (Kummer, 1836).

- Kümmer's third summation theorem [(Kummer, 1836). p.134]:

$${}_2F_1\left[\begin{matrix} a, 1-a; \\ \\ c; \end{matrix} \; \frac{1}{2}\right] = \frac{2^{1-c}\,\Gamma(c)\,\Gamma(1/2)}{\Gamma\left(\frac{c+a}{2}\right)\Gamma\left(\frac{1+c-a}{2}\right)} \qquad (1.4)$$

$$= \frac{\Gamma\left(\frac{c}{2}\right)\Gamma\left(\frac{c+1}{2}\right)}{\Gamma\left(\frac{c+a}{2}\right)\Gamma\left(\frac{1+c-a}{2}\right)} \qquad (1.5)$$

where $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

In the literature, the above summation formula is also known as "Bailey's summation theorem".

- Summation theorem given by (Rakha & Rathie, 2011), p.828, Theorem (6)):

$$
{}_2F_1\left[\begin{matrix}\alpha,1-\alpha+m; & \\ & \dfrac{1}{2}\\ \beta; & \end{matrix}\right]
= \frac{2^{1+m-\beta}\sqrt{\pi}\;\Gamma(a+m)\Gamma(\beta)}{\Gamma(\alpha)\Gamma\left(\dfrac{\beta-\alpha}{2}\right)\Gamma\left(\dfrac{\beta-\alpha+1}{2}\right)}\times
\sum_{n=0}^{m}\left\{(-1)^n\binom{m}{n}\frac{\Gamma\left(\dfrac{\beta-\alpha+n}{2}\right)}{\Gamma\left(\left(\dfrac{\beta+\alpha+n}{2}\right)-m\right)}\right\},
\tag{1.6}
$$

where $\alpha,\beta,\beta-\alpha,\alpha-m\in\mathbb{C}\setminus\mathbb{Z}_0^-$ and $m\in\mathbb{N}_0$.

Motivated by the work collected in the beautiful monographs of (Andrews et al., 1999; Bailey, 1953; Carlson, 1977; Erdélyi et al., 1955; Prudnikov et al., 1986; Slater, 1966; Srivastava & Choi, 2011) and the papers of (Arora & Singh, 2008; Kim et al., 2010; Kim et al., 2013; Miller, 2005; Qureshi & Baboo, 2016; Qureshi & Khan, 2020), and others (Awad et al., 2021; Koepf et al., 2019), we are interested in giving some summation formulas for ${}_3F_2[1/2]$ in Section 2. In Section 3 and 4, we have given some summation formulas for ${}_4F_3[1/2]$ and ${}_5F_4[1/2]$ respectively. The detailed proof of summation formulas has been provided by using the summation theorem given by Rakha-Rathie and the series rearrangement technique.

Any values of the numerator and denominator parameters in sections 2, 3, and 4, leading to results which do not make sense are tacitly excluded.

2- Summation formulas for ${}_3F_2[1/2]$

Theorem 2.1. *The following summation theorem holds true*:

$$
{}_3F_2\left[\begin{matrix}a,1-a+p,c+1; & \\ & \dfrac{1}{2}\\ b,c; & \end{matrix}\right]
= \frac{2^{1+p-b}\sqrt{\pi}\;\Gamma(b)}{\Gamma(a)\Gamma\left(\dfrac{b-a}{2}\right)\Gamma\left(\dfrac{b-a+1}{2}\right)}\times
\left[\Gamma(a-p)\sum_{r=0}^{p}\left\{(-1)^r\binom{p}{r}\frac{\Gamma\left(\dfrac{b-a+r}{2}\right)}{\Gamma\left(\dfrac{b+a+r-2p}{2}\right)}\right\}+
+\frac{(1-a+p)\Gamma(a-p-1)}{c}\times
\sum_{r=0}^{p+2}\left\{(-1)^r\binom{p+2}{r}\frac{\Gamma\left(\dfrac{b-a+r}{2}\right)}{\Gamma\left(\dfrac{b+a+r-2p-2}{2}\right)}\right\}\right],
\tag{2.1}
$$

where $a,b,c,b-a,a-p\in\mathbb{C}\setminus\mathbb{Z}_0^-$ and $p\in\mathbb{N}_0$.

Proof of Theorem (2.1): In order to establish the result, we proceed as follows:

$$
{}_3F_2\left[\begin{matrix}a,1-a+p,c+1; & \\ & \dfrac{1}{2}\\ b,c; & \end{matrix}\right] =
$$

$$
= \sum_{r=0}^{\infty}\frac{(a)_r(1-a+p)_r(c+1)_r(1/2)^r}{(b)_r(c)_r\,r!}
$$

$$
= \sum_{r=0}^{\infty}\frac{(a)_r(1-a+p)_r(1/2)^r}{(b)_r\,r!}\left\{1+\frac{r}{c}\right\}
$$

$$
=\sum_{r=0}^{\infty}\frac{(a)_r(1-a+p)_r(1/2)^r}{(b)_r\,r!}+\frac{1}{c}\sum_{r=1}^{\infty}\frac{(a)_r(1-a+p)_r(1/2)^r}{(b)_r(r-1)!}
\tag{2.2}
$$

Replacing r by $r+1$ in the second term on the right hand side of the equation (2.2), we get:

$$
{}_3F_2\left[\begin{matrix} a, 1-a+p, c+1; \\ \\ b, c; \end{matrix}\ \frac{1}{2}\right]
$$

$$
=\sum_{r=0}^{\infty}\frac{(a)_r(1-a+p)_r(1/2)^r}{(b)_r\, r!}+\frac{1}{c}\sum_{r=0}^{\infty}\frac{(a)_{r+1}(1-a+p)_{r+1}(1/2)^{r+1}}{(b)_{r+1}\, r!}
$$

$$
=\sum_{r=0}^{\infty}\frac{(a)_r(1-a+p)_r(1/2)^r}{(b)_r\, r!}+\frac{a(1-a+p)}{2bc}\sum_{r=0}^{\infty}\frac{(1+a)_r(2-a+p)_r(1/2)^r}{(1+b)_r\, r!}
$$

$$
={}_2F_1\left[\begin{matrix} a, 1-a+p\ ; \\ \\ b; \end{matrix}\ \frac{1}{2}\right]+\frac{a(1-a+p)}{2bc}\,{}_2F_1\left[\begin{matrix} 1+a,\ 2-a+p\ ; \\ \\ 1+b; \end{matrix}\ \frac{1}{2}\right]
$$

(2.3) Applying summation theorem (1.6) in equation (2.3), we get

$$
{}_3F_2\left[\begin{matrix} a, 1-a+p, c+1; \\ \\ b, c; \end{matrix}\ \frac{1}{2}\right]=
$$

$$
=\frac{2^{1+p-b}\sqrt{\pi}\ \Gamma(b)\Gamma(a-p)}{\Gamma(a)\Gamma\left(\frac{b-a}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)}\times\sum_{r=0}^{p}\left\{(-1)^r\binom{p}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\left(\frac{b+a+r}{2}\right)-p\right)}\right\}+
$$

$$
+\frac{2^{1+p-b}(1-a+p)\sqrt{\pi}\ \Gamma(b)\Gamma(a-p-1)}{c\Gamma(a)\Gamma\left(\frac{b-a}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)}\times\sum_{r=0}^{p+2}\left\{(-1)^r\binom{p+2}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\left(\frac{b+a+r+2}{2}\right)-(p+2)\right)}\right\},
$$

(2.4) On simplifying further, we arrive at the result (2.1).

Theorem 2.2. *The following summation theorem holds true*:

$$
{}_3F_2\left[\begin{matrix} a, 1-a+p, c+2; \\ \\ b,\ c; \end{matrix}\ \frac{1}{2}\right]
=\frac{2^{1+p-b}\sqrt{\pi}\ \Gamma(b)}{\Gamma(a)\Gamma\left(\frac{b-a}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)}\times
$$

$$
\times\left[\Gamma(a-p)\sum_{r=0}^{p}\left\{(-1)^r\binom{p}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p}{2}\right)}\right\}+\right.
$$

$$
+\frac{2(1-a+p)\Gamma(a-p-1)}{c}\times\sum_{r=0}^{p+2}\left\{(-1)^r\binom{p+2}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-2}{2}\right)}\right\}+
$$

$$
\left.+\frac{(1-a+p)(2-a+p)\Gamma(a-p-2)}{c(c+1)}\times\sum_{r=0}^{p+4}\left\{(-1)^r\binom{p+4}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-4}{2}\right)}\right\}\right],
$$

(2.5)

where $a, b, c, b-a, a-p \in \mathbb{C}\setminus\mathbb{Z}_0^-$ and $p\in\mathbb{N}_0$.

Theorem 2.3. *The following summation theorem holds true:*

$$
{}_3F_2\left[\begin{matrix} a, 1-a+p, c+3; \\ \\ b, c; \end{matrix}\ \frac{1}{2}\right] = \frac{2^{1+p-b}\sqrt{\pi}\ \Gamma(b)}{\Gamma(a)\Gamma\left(\frac{b-a}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)} \times
$$

$$
\times\left[\Gamma(a-p)\sum_{r=0}^{p}\left\{(-1)^r\binom{p}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p}{2}\right)}\right\} + \right.
$$

$$
+\frac{3(1-a+p)\Gamma(a-p-1)}{c}\times
$$

$$
\times\sum_{r=0}^{p+2}\left\{(-1)^r\binom{p+2}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-2}{2}\right)}\right\} +
$$

$$
+\frac{3(1-a+p)(2-a+p)\Gamma(a-p-2)}{c(c+1)}\times
$$

$$
\times\sum_{r=0}^{p+4}\left\{(-1)^r\binom{p+4}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-4}{2}\right)}\right\} +
$$

$$
+\frac{(1-a+p)(2-a+p)(3-a+p)\Gamma(a-p-3)}{c(c+1)(c+2)}\times
$$

$$
\left.\times\sum_{r=0}^{p+6}\left\{(-1)^r\binom{p+6}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-6}{2}\right)}\right\}\right],
$$

(2.6)

where $a, b, c, b-a, a-p \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $p \in \mathbb{N}_0$.

Theorem 2.4. The following formula holds true:

$$
{}_3F_2\left[\begin{matrix} a, 1-a+p, c+4; \\ \\ b, c; \end{matrix}\ \frac{1}{2}\right] = \frac{2^{1+p-b}\sqrt{\pi}\ \Gamma(b)}{\Gamma(a)\Gamma\left(\frac{b-a}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)} \times
$$

$$
\times\left[\Gamma(a-p)\sum_{r=0}^{p}\left\{(-1)^r\binom{p}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p}{2}\right)}\right\} + \right.
$$

$$
+\frac{4(1-a+p)\Gamma(a-p-1)}{c}\times
$$

$$
\times\sum_{r=0}^{p+2}\left\{(-1)^r\binom{p+2}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-2}{2}\right)}\right\} +
$$

$$
+\frac{6(1-a+p)(2-a+p)\Gamma(a-p-2)}{c(c+1)}\times
$$

$$
\times\sum_{r=0}^{p+4}\left\{(-1)^r\binom{p+4}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-4}{2}\right)}\right\} +
$$

$$
+\frac{4(1-a+p)(2-a+p)(3-a+p)\Gamma(a-p-3)}{c(c+1)(c+2)}\times
$$

$$
\times\sum_{r=0}^{p+6}\left\{(-1)^r\binom{p+6}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-6}{2}\right)}\right\} +
$$

$$
+\frac{(1-a+p)(2-a+p)(3-a+p)(4-a+p)\Gamma(a-p-4)}{c(c+1)(c+2)(c+3)}\times
$$

$$
\left.\times\sum_{r=0}^{p+8}\left\{(-1)^r\binom{p+8}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-8}{2}\right)}\right\}\right], \quad (2.7)
$$

where $a, b, c, b-a, a-p \in \mathbb{C}\setminus\mathbb{Z}_0^-$ and $p\in\mathbb{N}_0$.

The proof of theorems (2.2)-(2.4), would run parallel to theorem (2.1) with the help of summation theorem (1.6) and the series rearrangement technique. The involved details are omitted.

3- Summation formulas for ${}_4F_3\left[\frac{1}{2}\right]$

Theorem 3.1. *The following summation theorem holds true*:

$$
{}_4F_3\left[\begin{matrix} a, 1-a+p, c+1, d+1; \\ & \frac{1}{2} \\ b, c, d; \end{matrix}\right] = \frac{2^{1+p-b}\sqrt{\pi}\ \Gamma(b)}{\Gamma(a)\Gamma\left(\frac{b-a}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)}\times
$$

$$
\times\left[\Gamma(a-p)\sum_{r=0}^{p}\left\{(-1)^r\binom{p}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p}{2}\right)}\right\}+\right.
$$

$$
+\frac{(1-a+p)(1+c+d)\Gamma(a-p-1)}{cd}\times
$$

$$
\times\sum_{r=0}^{p+2}\left\{(-1)^r\binom{p+2}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-2}{2}\right)}\right\}+
$$

$$
+\frac{(1-a+p)(2-a+p)\Gamma(a-p-2)}{cd}\times
$$

$$
\left.\times\sum_{r=0}^{p+4}\left\{(-1)^r\binom{p+4}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-4}{2}\right)}\right\}\right], \tag{3.1}
$$

where $a, b, c, d, b-a, a-p \in \mathbb{C}\setminus\mathbb{Z}_0^-$ and $p\in\mathbb{N}_0$.

Proof of the Theorem (3.1):

In order to establish the result, we proceed as follows.

$$
{}_4F_3\left[\begin{matrix} a, 1-a+p, c+1, d+1; \\ & \frac{1}{2} \\ b, c, d; \end{matrix}\right] =
$$

$$
= \sum_{r=0}^{\infty}\frac{(a)_r(1-a+p)_r(c+1)_r(d+1)(1/2)^r}{(b)_r(c)_r(d)_r\, r!}
$$

$$
= \sum_{r=0}^{\infty}\frac{(a)_r(1-a+p)_r(1/2)^r}{(b)_r\, r!}\left\{1+\frac{(1+c+d)r}{cd}+\frac{r(r-1)}{cd}\right\}
$$

$$
= \sum_{r=0}^{\infty}\frac{(a)_r(1-a+p)_r(1/2)^r}{(b)_r\, r!}+\frac{(1+c+d)}{cd}\times
$$

$$
\times\sum_{r=1}^{\infty}\frac{(a)_r(1-a+p)_r(1/2)^r}{(b)_r(r-1)!}+
$$

$$
+\frac{1}{cd}\sum_{r=2}^{\infty}\frac{(a)_r(1-a+p)_r(1/2)^r}{(b)_r(r-2)!} \tag{3.2}
$$

Replacing r by $r+1$ in the second term and r by $r+2$ in the third term on the right hand side of the equation (3.2), we get

$$
{}_4F_3\left[\begin{matrix} a, 1-a+p, c+1, d+1; \\ & \frac{1}{2} \\ b, c, d; \end{matrix}\right] =
$$

$$
= \sum_{r=0}^{\infty}\frac{(a)_r(1-a+p)_r(1/2)^r}{(b)_r\, r!}+\frac{(1+c+d)}{cd}\times
$$

$$
\times\sum_{r=0}^{\infty}\frac{(a)_{r+1}(1-a+p)_{r+1}(1/2)^{r+1}}{(b)_{r+1}\, r!}+
$$

$$
+\frac{1}{cd}\sum_{r=0}^{\infty}\frac{(a)_{r+2}(1-a+p)_{r+2}(1/2)^{r+2}}{(b)_{r+2}\, r!}
$$

$$
= \sum_{r=0}^{\infty}\frac{(a)_r(1-a+p)_r(1/2)^r}{(b)_r\, r!}+\frac{(1+c+d)a(1-a+p)}{2bcd}\times
$$

$$
\times\sum_{r=0}^{\infty}\frac{(1+a)_r(2-a+p)_r(1/2)^r}{(1+b)_r\, r!}+
$$

$$
+\frac{a(a+1)(1-a+p)(2-a+p)}{4b(b+1)cd}\times
$$

$$\times\sum_{r=0}^{\infty}\frac{(2+a)_r(3-a+p)_r(1/2)^r}{(2+b)_r\,r!}$$

$$= {}_2F_1\left[\begin{matrix}a,1-a+p\,;\\ \qquad\frac{1}{2}\\ b;\end{matrix}\right]+\frac{(1+c+d)a(1-a+p)}{2bcd}\times$$

$$\times{}_2F_1\left[\begin{matrix}1+a,\,2-a+p\,;\\ \qquad\frac{1}{2}\\ 1+b;\end{matrix}\right]+$$

$$+\frac{a(a+1)(1-a+p)(2-a+p)}{4b(b+1)cd}\times$$

$$\times{}_2F_1\left[\begin{matrix}2+a,\,3-a+p\,;\\ \qquad\frac{1}{2}\\ 2+b;\end{matrix}\right]. \qquad (3.3)$$

Now applying the summation theorem (1.6) in the equation (3.3), we get

$${}_4F_3\left[\begin{matrix}a,1-a+p,\,c+1,\,d+1;\\ \qquad\frac{1}{2}\\ b,\,c,\,d\,;\end{matrix}\right]=$$

$$=\frac{2^{1+p-b}\sqrt{\pi}\;\Gamma(b)\Gamma(a-p)}{\Gamma(a)\Gamma\left(\frac{b-a}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)}\times$$

$$\times\sum_{r=0}^{p}\left\{(-1)^r\binom{p}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\left(\frac{b+a+r}{2}\right)-p\right)}\right\}+$$

$$+\frac{2^{1+p-b}(1-a+p)(1+c+d)\sqrt{\pi}\;\Gamma(b)\Gamma(a-p-1)}{c\,d\,\Gamma(a)\Gamma\left(\frac{b-a}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)}\times$$

$$\times\sum_{r=0}^{p+2}\left\{(-1)^r\binom{p+2}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\left(\frac{b+a+r+2}{2}\right)-(p+2)\right)}\right\}+$$

$$+\frac{2^{1+p-b}(1-a+p)(2-a+p)\sqrt{\pi}\,\Gamma(b)\Gamma(a-p-2)}{cd\,\Gamma(a)\Gamma\left(\frac{b-a}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)}\times$$

$$\times\sum_{r=0}^{p+4}\left\{(-1)^r\binom{p+4}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\left(\frac{b+a+r+4}{2}\right)-(p+4)\right)}\right\}.$$

(3.4) On simplifying further, we arrive at the result (3.1) .

Theorem 3.2. *The following summation theorem holds true*:

$${}_4F_3\left[\begin{matrix}a,1-a+p,\,c+1,\,d+2;\\ \qquad\frac{1}{2}\\ b,\,c,\,d\,;\end{matrix}\right]$$

$$=\frac{2^{1+p-b}\sqrt{\pi}\;\Gamma(b)}{\Gamma(a)\Gamma\left(\frac{b-a}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)}\times$$

$$\times\left[\Gamma(a-p)\sum_{r=0}^{p}\left\{(-1)^r\binom{p}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p}{2}\right)}\right\}+\right.$$

$$+\frac{(1-a+p)(2+2c+d)\Gamma(a-p-1)}{cd}\times$$

$$\times\sum_{r=0}^{p+2}\left\{(-1)^r\binom{p+2}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-2}{2}\right)}\right\}+$$

$$+\frac{(1-a+p)(2-a+p)(4+c+2d)\Gamma(a-p-2)}{cd(d+1)}\times$$

$$\times\sum_{r=0}^{p+4}\left\{(-1)^r\binom{p+4}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-4}{2}\right)}\right\}+$$

$$+\frac{(1-a+p)(2-a+p)(3-a+p)\Gamma(a-p-3)}{cd(d+1)}\times$$

$$\times \sum_{r=0}^{p+6}\left\{(-1)^r \binom{p+6}{r} \frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-6}{2}\right)}\right\}\Bigg]\Bigg], \tag{3.5}$$

where $a, b, c, d, b-a, a-p \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $p \in \mathbb{N}_0$.

Theorem 3.3. *The following summation theorem holds true*:

$$ {}_4F_3\left[\begin{matrix} a, 1-a+p, c+2, d+2; & \\ & \frac{1}{2} \\ b, c, d; & \end{matrix}\right]$$

$$= \frac{2^{1+p-b}\sqrt{\pi}\ \Gamma(b)}{\Gamma(a)\Gamma\left(\frac{b-a}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)} \times$$

$$\times\Bigg[\Gamma(a-p)\sum_{r=0}^{p}\left\{(-1)^r \binom{p}{r} \frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p}{2}\right)}\right\} +$$

$$+ \frac{(1-a+p)(4+2c+2d)\Gamma(a-p-1)}{cd} \times$$

$$\times \sum_{r=0}^{p+2}\left\{(-1)^r \binom{p+2}{r} \frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-2}{2}\right)}\right\} +$$

$$+ \frac{(1-a+p)(2-a+p)\Gamma(a-p-2)}{c(c+1)} \times$$

$$\times \frac{(14+c^2+d^2+4cd+9c+9d)}{d(d+1)} \times$$

$$\times \sum_{r=0}^{p+4}\left\{(-1)^r \binom{p+4}{r} \frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-4}{2}\right)}\right\} +$$

$$+ \frac{(1-a+p)(2-a+p)(3-a+p)}{c(c+1)} \times$$

$$\times \frac{(8+2c+2d)\Gamma(a-p-3)}{d(d+1)} \times$$

$$\times \sum_{r=0}^{p+6}\left\{(-1)^r \binom{p+6}{r} \frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-6}{2}\right)}\right\} +$$

$$+ \frac{(1-a+p)(2-a+p)(3-a+p)}{c(c+1)} \times$$

$$\times \frac{(4-a+p)\Gamma(a-p-4)}{d(d+1)} \times$$

$$\times \sum_{r=0}^{p+8}\left\{(-1)^r \binom{p+8}{r} \frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-8}{2}\right)}\right\}\Bigg]\Bigg], \tag{3.6}$$

where $a, b, c, d, b-a, a-p \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $p \in \mathbb{N}_0$.

Theorem 3.4. *The following summation theorem holds true*:

$$ {}_4F_3\left[\begin{matrix} a, 1-a+p, c+1, d+3; & \\ & \frac{1}{2} \\ b, c, d; & \end{matrix}\right]$$

$$= \frac{2^{1+p-b}\sqrt{\pi}\ \Gamma(b)}{\Gamma(a)\Gamma\left(\frac{b-a}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)} \times$$

$$\times\Bigg[\Gamma(a-p)\sum_{r=0}^{p}\left\{(-1)^r \binom{p}{r} \frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p}{2}\right)}\right\} +$$

$$+ \frac{(3+3c+d)(1-a+p)\Gamma(a-p-1)}{cd} \times$$

$$\times \sum_{r=0}^{p+2}\left\{(-1)^r \binom{p+2}{r} \frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-2}{2}\right)}\right\} +$$

$$+\frac{(9+3c+3d)(1-a+p)(2-a+p)\Gamma(a-p-2)}{cd(d+1)}\times$$
$$\times\sum_{r=0}^{p+4}\left\{(-1)^r\binom{p+4}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-4}{2}\right)}\right\}+$$
$$+\frac{(1-a+p)(2-a+p)(3-a+p)(9+c+3d)\Gamma(a-p-3)}{cd(d+1)(d+2)}\times$$
$$\times\sum_{r=0}^{p+6}\left\{(-1)^r\binom{p+6}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-6}{2}\right)}\right\}+$$
$$+\frac{(1-a+p)(2-a+p)(3-a+p)(4-a+p)\Gamma(a-p-4)}{cd(d+1)(d+2)}\times$$
$$\times\sum_{r=0}^{p+8}\left\{(-1)^r\binom{p+8}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-8}{2}\right)}\right\}\Bigg],$$
(3.7)

where $a,b,c,d,b-a,a-p\in\mathbb{C}\setminus\mathbb{Z}_0^-$ and $p\in\mathbb{N}_0$.

The proof of theorems (3.2)-(3.4) would be accomplished by following the lines of that of theorem (3.1) with the aid of summation theorem (1.6). The involved details are omitted.

4- Summation formulas for ${}_5F_4\left[\frac{1}{2}\right]$ and ${}_6F_5\left[\frac{1}{2}\right]$

Theorem 4.1. *The following summation theorem holds true*:

$${}_5F_4\left[\begin{matrix}a,1-a+p,\ c+1,\ d+1,\ g+1;\\ \\ b,\ c,\ d,\ g;\end{matrix}\ \frac{1}{2}\right]$$
$$=\frac{2^{1+p-b}\sqrt{\pi}\ \Gamma(b)}{\Gamma(a)\Gamma\left(\frac{b-a}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)}\times$$
$$\times\Bigg[\Gamma(a-p)\sum_{r=0}^{p}\left\{(-1)^r\binom{p}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p}{2}\right)}\right\}+$$
$$+\frac{(1+c+d+g+cd+cg+dg)}{cdg}\times$$
$$\times(1-a+p)\Gamma(a-p-1)\times$$
$$\times\sum_{r=0}^{p+2}\left\{(-1)^r\binom{p+2}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-2}{2}\right)}\right\}+$$
$$+\frac{(3+c+d+g)(1-a+p)(2-a+p)\Gamma(a-p-2)}{cdg}\times$$
$$\times\sum_{r=0}^{p+4}\left\{(-1)^r\binom{p+4}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-4}{2}\right)}\right\}+$$
$$+\frac{(1-a+p)(2-a+p)(3-a+p)\Gamma(a-p-3)}{cdg}\times$$
$$\times\sum_{r=0}^{p+6}\left\{(-1)^r\binom{p+6}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-6}{2}\right)}\right\}\Bigg],$$
(4.1)

where $a,b,c,d,g,b-a,a-p\in\mathbb{C}\setminus\mathbb{Z}_0^-$ and $p\in\mathbb{N}_0$.

Proof of Theorem 4.1.**:** In order to establish the result, we proceed as follows.

$${}_5F_4\left[\begin{matrix}a,1-a+p,\ c+1,\ d+1,\ g+1;\\ \\ b,\ c,\ d,\ g;\end{matrix}\ \frac{1}{2}\right]$$
$$=\sum_{r=0}^{\infty}\frac{(a)_r(1-a+p)_r(c+1)_r(d+1)_r}{(b)_r(c)_r(d)_r}\times$$

$$\times \frac{(g+1)_r (1/2)^r}{(g)_r \, r!}$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (1-a+p)_r (1/2)^r}{(b)_r \, r!} \times$$

$$\times \left\{ 1 + \frac{(1+c+d+g+cd+cg+dg) r}{cdg} + \right.$$

$$+ \frac{(3+c+d+g)}{cdg} r(r-1) +$$

$$\left. + \frac{1}{cdg} r(r-1)(r-2) \right\}$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (1-a+p)_r (1/2)^r}{(b)_r \, r!} +$$

$$+ \frac{(1+c+d+g+cd+cg+dg)}{cdg} \times$$

$$\times \sum_{r=1}^{\infty} \frac{(a)_r (1-a+p)_r (1/2)^r}{(b)_r \, (r-1)!} +$$

$$+ \frac{(3+c+d+g)}{cdg} \sum_{r=2}^{\infty} \frac{(a)_r (1-a+p)_r (1/2)^r}{(b)_r \, (r-2)!} +$$

$$+ \frac{1}{cdg} \sum_{r=3}^{\infty} \frac{(a)_r (1-a+p)_r (1/2)^r}{(b)_r \, (r-3)!}. \quad (4.2)$$

Replacing r by $r+1$ in the second term, r by $r+2$ in the third term, and r by $r+3$ in the fourth term on the right hand side of the equation (4.2), we get

$${}_5F_4 \left[\begin{matrix} a, 1-a+p, c+1, d+1, g+1; \\ \\ b, c, d, g; \end{matrix} \; \frac{1}{2} \right]$$

$$= \sum_{r=0}^{\infty} \frac{(a)_{r+1} (1-a+p)_{r+1} (1/2)^{r+1}}{(b)_{r+1} \, r!} +$$

$$+ \frac{(1+c+d+g+cd+cg+dg)}{cdg} \times$$

$$\times \sum_{r=0}^{\infty} \frac{(a)_{r+1} (1-a+p)_{r+1} (1/2)^{r+1}}{(b)_{r+1} \, r!} +$$

$$+ \frac{(1+c+d+g+cd+cg+dg)}{cdg} \times$$

$$+ \frac{(3+c+d+g)}{cdg} \sum_{r=0}^{\infty} \frac{(a)_{r+2} (1-a+p)_{r+2} (1/2)^{r+1}}{(b)_{r+2} \, r!} +$$

$$+ \frac{1}{cdg} \sum_{r=0}^{\infty} \frac{(a)_{r+3} (1-a+p)_{r+3} (1/2)^{r+3}}{(b)_{r+3} \, r!}.$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (1-a+p)_r (1/2)^r}{(b)_r \, r!} +$$

$$+ \frac{(1+c+d+g+cd+cg+dg) a (1-a+p)}{2bcdg} \times$$

$$\times \sum_{r=0}^{\infty} \frac{(1+a)_r (2-a+p)_r (1/2)^r}{(1+b)_r \, r!} +$$

$$+ \frac{(3+c+d+g) a (1+a)(1-a+p)(2-a+p)}{4b(b+1)cdg} \times$$

$$\times \sum_{r=0}^{\infty} \frac{(2+a)_r (3-a+p)_r (1/2)^r}{(2+b)_r \, r!} +$$

$$+ \frac{a(1+a)(1-a+p)(2-a+p)(3-a+p)}{8b(b+1)(b+2)cdg} \times$$

$$\times \sum_{r=0}^{\infty} \frac{(3+a)_r (4-a+p)_r (1/2)^r}{(3+b)_r \, r!}$$

$$= {}_2F_1 \left[\begin{matrix} a, 1-a+p; \\ \\ b; \end{matrix} \; \frac{1}{2} \right] +$$

$$+ \frac{(1+c+d+g+cd+cg+dg) a (1-a+p)}{2bcdg} \times$$

$$= {}_2F_1 \left[\begin{matrix} 1+a, 2-a+p; \\ \\ 1+b; \end{matrix} \; \frac{1}{2} \right] +$$

$$+ \frac{(3+c+d+g) a (1+a)(1-a+p)(2-a+p)}{4b(b+1)cdg} \times$$

$$= {}_2F_1\left[\begin{matrix} 2+a, 3-a+p; & \\ & \frac{1}{2} \\ 2+b; & \end{matrix}\right] +$$

$$+\frac{a(1+a)(1-a+p)(2-a+p)(3-a+p)}{8b(b+1)(b+2)cdg}\times$$

$$= {}_2F_1\left[\begin{matrix} 3+a, 4-a+p; & \\ & \frac{1}{2} \\ 3+b; & \end{matrix}\right] \qquad (4.3)$$

Now applying the summation theorem (1.6) in equation (4.3), we get

$${}_5F_4\left[\begin{matrix} a, 1-a+p, c+1, d+1, g+1; & \\ & \frac{1}{2} \\ b, c, d, g; & \end{matrix}\right]$$

$$= \frac{2^{1+p-b}\sqrt{\pi}\ \Gamma(b)\Gamma(a-p)}{\Gamma(a)\Gamma\left(\frac{b-a}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)}\times$$

$$\times\sum_{r=0}^{p}\left\{(-1)^r\binom{p}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r}{2}-p\right)}\right\}+$$

$$+\frac{2^{1+p-b}(1-a+p)(1+c+d+g+cd+cg+dg)}{cdg\ \Gamma(a)}\times$$

$$\times\frac{\sqrt{\pi}\,\Gamma(b)\Gamma(a-p-1)}{\Gamma\left(\frac{b-a}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)}\times$$

$$\times\sum_{r=0}^{p+2}\left\{(-1)^r\binom{p+2}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\left(\frac{b+a+r+2}{2}\right)-(p+2)\right)}\right\}+$$

$$+\frac{2^{1+p-b}(1-a+p)(2-a+p)(3+c+d+g)}{cdg\ \Gamma(a)\Gamma\left(\frac{b-a}{2}\right)}\times$$

$$\times\frac{\sqrt{\pi}\ \Gamma(b)\Gamma(a-p-2)}{\Gamma\left(\frac{b-a+1}{2}\right)}\times$$

$$\times\sum_{r=0}^{p+4}\left\{(-1)^r\binom{p+4}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\left(\frac{b+a+r+4}{2}\right)-(p+4)\right)}\right\}+$$

$$+\frac{2^{1+p-b}(1-a+p)(2-a+p)}{cdg\ \Gamma(a)\Gamma\left(\frac{b-a}{2}\right)}\times$$

$$\times\frac{(3-a-p)\Gamma(a-p-3)\sqrt{\pi}\ \Gamma(b)}{\Gamma\left(\frac{b-a+1}{2}\right)}\times$$

$$\times\sum_{r=0}^{p+6}\left\{(-1)^r\binom{p+6}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\left(\frac{b+a+r+6}{2}\right)-(p+6)\right)}\right\}.$$

On simplifying further, we arrive at the result (4.1).

Theorem 4.2. *The following summation theorem holds true*:

:

$${}_5F_4\left[\begin{matrix} a, 1-a+p, c+1, d+1, g+2; & \\ & \frac{1}{2} \\ b, c, d, g; & \end{matrix}\right]$$

$$= \frac{2^{1+p-b}\sqrt{\pi}\ \Gamma(b)}{\Gamma(a)\Gamma\left(\frac{b-a}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)}\times$$

$$\times\left[\Gamma(a-p)\sum_{r=0}^{p}\left\{(-1)^r\binom{p}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p}{2}\right)}\right\}+\right.$$

$$+\frac{(dg+cg+2cd+2c+2d+g+2)}{cdg}\times$$

$$\times(1-a+p)\Gamma(a-p-1)\times$$

$$\times\sum_{r=0}^{p+2}\left\{(-1)^r\binom{p+2}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-2}{2}\right)}\right\}+$$

$$+\frac{(1-a+p)(2-a+p)}{cdg(g+1)}\times$$

$$\times(10+cd+2dg+2cg+g^2+4c+4d+7g)\times$$

$$\times\Gamma(a-p-2)\sum_{r=0}^{p+4}\left\{(-1)^r\binom{p+4}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-4}{2}\right)}\right\}+$$

$$+\frac{(1-a+p)(2-a+p)(3-a+p)}{c\,d}\times$$

$$\times\frac{(7+c+d+2g)\Gamma(a-p-3)}{g(g+1)}\times$$

$$\times\left\{\sum_{r=0}^{p+6}(-1)^r\binom{p+6}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-6}{2}\right)}\right\}+$$

$$+\frac{(1-a+p)(2-a+p)(3-a+p)(4-a+p)\Gamma(a-p-4)}{c\,d\,g(g+1)}\times$$

$$\times\sum_{r=0}^{p+8}\left\{(-1)^r\binom{p+8}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-8}{2}\right)}\right\}\Bigg]\Bigg], \qquad (4.4)$$

where $a,b,c,d,g,\,b-a,\,a-p\in\mathbb{C}\setminus\mathbb{Z}_0^-$ and $p\in\mathbb{N}_0$.

The proof of theorem (4.2) would run parallel to the theorem (4.1) with the help of summation theorem (1.6). The details are omitted.

Theorem 4.3. *The following summation theorem holds true*:

$${}_6F_5\left[\begin{matrix}a,1-a+p,\,c+1,\,d+1,\,g+1,h+1; \\ \\ b,\,c,d,g,h;\end{matrix}\;\frac{1}{2}\right]$$

$$=\frac{2^{1+p-b}\sqrt{\pi}\;\Gamma(b)}{\Gamma(a)\Gamma\left(\frac{b-a}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)}\times$$

$$\times\left[\Gamma(a-p)\sum_{r=0}^{p}\left\{(-1)^r\binom{p}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p}{2}\right)}\right\}+\right.$$

$$+\frac{(1-a+p)\Gamma(a-p-1)(1+cdg+cdh+cgh+dgh+cd+cg+ch+dg+dh+gh+c+d+g+h)}{cd\,gh}\times$$

$$\times\sum_{r=0}^{p+2}\left\{(-1)^r\binom{p+2}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-2}{2}\right)}\right\}+$$

$$+\frac{(1-a+p)(2-a+p)(7+cd+cg+ch+dg+dh+gh+3c+3d+3g+3h)\Gamma(a-p-2)}{cd\,gh}\times$$

$$\times\sum_{r=0}^{p+4}\left\{(-1)^r\binom{p+4}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-4}{2}\right)}\right\}+$$

$$\times\frac{(6+c+d+g+h)(1-a+p)(2-a+p)(3-a+p)}{cdgh}\times$$

$$\times\Gamma(a-p-3)\sum_{r=0}^{p+6}\left\{(-1)^r\binom{p+6}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-6}{2}\right)}\right\}+$$

$$+\frac{(1-a+p)(2-a+p)(3-a+p)(4-a+p)\Gamma(a-p-4)}{cdgh}\times$$

$$\times\sum_{r=0}^{p+8}\left\{(-1)^r\binom{p+8}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\frac{b+a+r-2p-8}{2}\right)}\right\}\Bigg]\Bigg],$$

(4.5) where $a,b,c,d,g,h,\,b-a,\,a-p\in\mathbb{C}\setminus\mathbb{Z}_0^-$ and $p\in\mathbb{N}_0$.

Proof of Theorem 4.3.**:** In order to establish the result, we proceed as follows.

$$ {}_6F_5\left[\begin{matrix} a, 1-a+p, c+1, d+1, g+1, h+1; \\ \\ b, c, d, g, h; \end{matrix} \; \frac{1}{2}\right] $$

$$ = \sum_{r=0}^{\infty} \frac{(a)_r (1-a+p)_r (c+1)_r (d+1)_r}{(b)_r (c)_r (d)_r} \times $$

$$ \times \frac{(g+1)_r (h+1)_r (1/2)^r}{(g)_r (h)_r \, r!} $$

$$ = \sum_{r=0}^{\infty} \frac{(a)_r (1-a+p)_r (1/2)^r}{(b)_r \, r!} \times $$

$$ \times \left\{ 1 + \frac{(1+cdg+cdh+cgh+dgh+cd+cg+}{cd} \right. $$

$$ \frac{+ch+dg+dh+gh+c+d+g+h)}{gh} r + $$

$$ + \frac{(7+cd+cg+ch+dg+dh+gh+}{cd} $$

$$ \frac{+3c+3d+3g+3h)}{gh} r(r-1) + $$

$$ + \frac{(6+c+d+g+h)}{cdgh} r(r-1)(r-2) + $$

$$ \left. + \frac{1}{cdgh} r(r-1)(r-2)(r-3) \right\} \qquad (4.6) $$

$$ = \sum_{r=0}^{\infty} \frac{(a)_r (1-a+p)_r (1/2)^r}{(b)_r \, r!} + $$

$$ + \frac{(1+cdg+cdh+cgh+dgh+cd+cg+}{cd} $$

$$ \frac{+ch+dg+dh+gh+c+d+g+h)}{gh} \times $$

$$ \times \sum_{r=1}^{\infty} \frac{(a)_r (1-a+p)_r (1/2)^r}{(b)_r (r-1)!} + $$

$$ + \frac{(7+cd+cg+ch+dg+dh+}{cd} $$

$$ \frac{+gh+3c+3d+3g+3h)}{gh} \times $$

$$ \times \sum_{r=2}^{\infty} \frac{(a)_r (1-a+p)_r (1/2)^r}{(b)_r (r-2)!} + $$

$$ + \frac{(6+c+d+g+h)}{cd\,gh} \sum_{r=3}^{\infty} \frac{(a)_r (1-a+p)_r (1/2)^r}{(b)_r (r-3)!} + $$

$$ + \frac{1}{cd\,g\,h} \sum_{r=4}^{\infty} \frac{(a)_r (1-a+p)_r (1/2)^r}{(b)_r (r-4)!} \qquad (4.7) $$

Replacing r by $r+1$ in the second term, r by $r+2$ in the third term, r by $r+3$ in the fourth term, and r by $r+4$ in the fifth term on the right hand side of the equation (4.7), we get

$$ {}_6F_5\left[\begin{matrix} a, 1-a+p, c+1, d+1, g+1, h+1; \\ \\ b, c, d, g, h; \end{matrix} \; \frac{1}{2}\right] $$

$$ = \sum_{r=0}^{\infty} \frac{(a)_r (1-a+p)_r (1/2)^r}{(b)_r \, r!} + $$

$$ + \frac{(1+cdg+cdh+cgh+dgh+cd+cg+}{cd} $$

$$ \frac{+ch+dg+dh+gh+c+d+g+h)\times}{gh} $$

$$ \frac{\times a(1-a+p)}{2b} \sum_{r=0}^{\infty} \frac{(1+a)_r (2-a+p)_r (1/2)^r}{(1+b)_r \, r!} + $$

$$ + \frac{(7+cd+cg+ch+dg+dh+gh+3c+3d+}{cdgh} $$

$$ \frac{+3g+3h)\, a(a+1)(1-a+p)(2-a-p)}{4b(b+1)} \times $$

$$ \times \sum_{r=0}^{\infty} \frac{(2+a)_r (3-a+p)_r (1/2)^r}{(2+b)_r \, r!} + $$

$$ + \frac{a(a+1)(a+2)(1-a+p)(2-a+p)}{8b(b+1)(b+2)} \times $$

$$ \times \frac{(3-a+p)(6+c+d+g+h)}{cdgh} \times $$

$$ \times \sum_{r=0}^{\infty} \frac{(3+a)_r (4-a+p)_r (1/2)^r}{(3+b)_r \, r!} + $$

$$ + \frac{a(a+1)(a+2)(a+3)(1-a+p)(2-a+p)}{16b(b+1)(b+2)(b+3)} \times $$

$$ \times \frac{(3-a+p)(4-a+p)}{cd\,gh} \sum_{r=0}^{\infty} \frac{(4+a)_r (5-a+p)_r (1/2)^r}{(4+b)_r \, r!} $$

$$= {}_2F_1\left[\begin{matrix} a, 1-a+p\,; \\ & \frac{1}{2} \\ b\,; \end{matrix}\right] +$$

$$+\frac{(1+cdg+cdh+cgh+dgh+cd+cg+ch+}{2cd}$$

$$\frac{+cg+ch+dg+dh+gh+c+d+g+h)}{2bgh}\times$$

$$\times a(1-a+b)\,{}_2F_1\left[\begin{matrix} 1+a,\ 2-a+p\,; \\ & \frac{1}{2} \\ 1+b\,; \end{matrix}\right] +$$

$$+\frac{(7+cd+cg+ch+dg+dh+gh+}{cd}$$

$$\frac{+3c+3d+3g+3h)a(a+1)(1-a+p)}{4b(b+1)gh}\times$$

$$\times\frac{a(a+1)(1-a+p)(2-a+p)}{4b(b+1)}\times$$

$$\times(2-a+p)\,{}_2F_1\left[\begin{matrix} 2+a,\ 3-a+p\,; \\ & \frac{1}{2} \\ 2+b\,; \end{matrix}\right] +$$

$$+\frac{a(a+1)(a+2)(1-a+p)(2-a+p)}{8b(b+1)(b+2)}\times$$

$$\times\frac{(3-a+p)(6+c+d+g+h)}{cdgh}\times$$

$$\times\,{}_2F_1\left[\begin{matrix} 3+a,\ 4-a+p\,; \\ & \frac{1}{2} \\ 3+b\,; \end{matrix}\right] +$$

$$+\frac{a(a+1)(a+2)(a+3)(1-a+p)(2-a+p)}{16b(b+1)(b+2)(b+3)}\times$$

$$\times\frac{(3-a+p)(4-a+p)}{cdgh}\,{}_2F_1\left[\begin{matrix} 4+a,\ 5-a+p\,; \\ & \frac{1}{2} \\ 4+b\,; \end{matrix}\right] \quad (4.8)$$

Applying the summation theorem (1.6) in equation (4.8), we get

$${}_6F_5\left[\begin{matrix} a, 1-a+p,\ c+1,\ d+1,\ g+1,\ h+1; \\ & \frac{1}{2} \\ b, c, d, g, h\,; \end{matrix}\right] =$$

$$= \frac{2^{1+p-b}\sqrt{\pi}\,\Gamma(b)\,\Gamma(a-p)}{\Gamma(a)\,\Gamma\left(\frac{b-a}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)}\times$$

$$\times\sum_{r=0}^{p}\left\{(-1)^r\binom{p}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\left(\frac{b+a+r}{2}\right)-p\right)}\right\}+$$

$$+\frac{2^{1+p-b}(1-a+p)\sqrt{\pi}\,\Gamma(b)\Gamma(a-p-1)}{\Gamma(a)\,\Gamma\left(\frac{b-a}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)}\times$$

$$\times\frac{(1+c\,dg+c\,dh+cgh+dgh+cd+cg+}{cd}$$

$$\frac{+ch+dg+dh+gh+c+d+g+h)}{gh}\times$$

$$\times\sum_{r=0}^{p+2}\left\{(-1)^r\binom{p+2}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\left(\frac{b+a+r+2}{2}\right)-(p+2)\right)}\right\}+$$

$$+\frac{2^{1+p-b}(1-a+p)(2-a+p)}{\Gamma(a)\Gamma\left(\frac{b-a}{2}\right)\Gamma\left(\frac{b-a+1}{2}\right)}\times$$

$$\times\frac{(7+cd+cg+ch+dg+dh+gh+3c+}{cd}$$

$$\frac{+3d+3g+3h)\sqrt{\pi}\,\Gamma(b)\,\Gamma(a-p-2)}{gh}\times$$

$$\times\sum_{r=0}^{p+4}\left\{(-1)^r\binom{p+4}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\left(\frac{b+a+r+4}{2}\right)-(p+4)\right)}\right\}+$$

$$+\frac{2^{1+p-b}(1-a+p)(2-a+p)(3-a+p)}{cdgh\,\Gamma(a)\Gamma\left(\frac{b-a}{2}\right)}\times$$

$$\times\frac{(6+c+d+g+h)\,\Gamma(a-p-3)\Gamma(b)\sqrt{\pi}}{\Gamma\left(\frac{b-a+1}{2}\right)}\times$$

$$\times\sum_{r=0}^{p+6}\left\{(-1)^r\binom{p+6}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\left(\frac{b+a+r+6}{2}\right)-(p+6)\right)}\right\}+$$

$$+\frac{2^{1+p-b}(1-a+p)(2-a+p)(3-a+p)}{cdgh\,\Gamma(a)\Gamma\left(\frac{b-a}{2}\right)}\times$$

$$\times\frac{(4-a+p)\Gamma(a-p-4)\Gamma(b)\sqrt{\pi}}{\Gamma\left(\frac{b-a+1}{2}\right)}\times$$

$$\times\sum_{r=0}^{p+8}\left\{(-1)^r\binom{p+8}{r}\frac{\Gamma\left(\frac{b-a+r}{2}\right)}{\Gamma\left(\left(\frac{b+a+r+8}{2}\right)-(p+8)\right)}\right\}.$$

On simplifying further, we arrive at the result (4.5).

CONCLUSION

In our present investigation, we have given certain extensions and generalizations of Kümmer's third summation Theorem (1.5) in the form of ${}_3F_2[1/2]$, ${}_4F_3[1/2]$, ${}_5F_4[1/2]$ and ${}_6F_5[1/2]$ where some numerator and denominator parameters differ by a positive integer, as claimed in the above theorems. We conclude this paper with the remark that many other summation theorems can be derived in an analogous manner. Moreover, the results deduced above are expected to lead to some potential applications in several fields of applied mathematics, statistics, and engineering sciences.

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